Poisson Regression

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The model

A count outcome, such as the number of bike rentals in an hour, is not well described by a normal distribution when the counts are small, since the normal distribution puts probability below zero, and a count is never below zero. A Poisson regression model takes the event rate to be the exponential function of a linear function of the covariates.

Definition 0.1 (Poisson regression model)**.** Let Y be a count outcome, $\underset{\sim}{X}$ a covariate vector, and T the exposure magnitude over which the events in Y are counted. A **Poisson regression model** assumes that, given $\underset{\sim}{X} = \underset{\sim}{x}$ and $T = t$, Y has a Poisson distribution with mean $\mu(\underset{\sim}{x}, t) = \lambda(\underset{\sim}{x}) \cdot t$, where the event rate $\lambda(\underset{\sim}{x})$ has a log that is a linear function of the covariates:

$$\ln(\lambda(\underset{\sim}{x})) = \beta_0 + \beta_1 x_1 + ... + \beta_p x_p.$$

Rate ratios

Definition 0.1 (Rate ratio)**.** In a Poisson regression model, the **rate ratio** comparing covariate pattern $\underset{\sim}{x}$ to covariate pattern $\underset{\sim}{x}^*$ is the ratio of their event rates:

$$_\lambda(\underset{\sim}{x} : \underset{\sim}{x}^*) \stackrel{\text{def}}{=} \frac{\lambda(\underset{\sim}{x})}{\lambda(\underset{\sim}{x}^*)}.$$

Theorem 0.1 (Rate ratios in Poisson regression)**.** *In a Poisson regression model, suppose covariate x_j enters the linear function $\ln(\lambda(\underset{\sim}{x}))$ only through the single term $\beta_j x_j$ (no interactions or transformations involving x_j). Let $\underset{\sim}{x}$ and $\underset{\sim}{x}^*$ be two covariate patterns*

that are identical except that $x_j = a$ and $x_j^* = b$. Then

$$\lambda(\underset{\sim}{x} : \underset{\sim}{x}^*) = \exp\{\beta_j(a-b)\} \tag{1}$$

and for any common exposure magnitude $t > 0$, the ratio of the means is the same rate ratio:

$$\frac{\mu(\underset{\sim}{x}, t)}{\mu(\underset{\sim}{x}^*, t)} = \lambda(\underset{\sim}{x} : \underset{\sim}{x}^*)$$

Proof

Proof. Because $\underset{\sim}{x}$ and $\underset{\sim}{x}^*$ agree on every covariate except x_j, every term of $\ln(\lambda(\underset{\sim}{x}))$ except $\beta_j x_j$ is shared with $\ln(\lambda(\underset{\sim}{x}^*))$:

$$\begin{aligned}
\ln(\lambda(\underset{\sim}{x})) - \ln(\lambda(\underset{\sim}{x}^*)) &= \big(\beta_0 + \beta_1 x_1 + \ldots + \beta_j a + \ldots + \beta_p x_p\big) \\
&\quad - \big(\beta_0 + \beta_1 x_1 + \ldots + \beta_j b + \ldots + \beta_p x_p\big) && \text{(substitute the linear function)} \\
&= \beta_j a - \beta_j b && \text{(cancel the shared terms)} \\
&= \beta_j(a-b) && \text{(factor out } \beta_j\text{)}
\end{aligned}$$

Then the rate ratio is:

$$\begin{aligned}
\lambda(\underset{\sim}{x} : \underset{\sim}{x}^*) &= \frac{\lambda(\underset{\sim}{x})}{\lambda(\underset{\sim}{x}^*)} && \text{(definition of rate ratio)} \\
&= \frac{\exp\{\ln(\lambda(\underset{\sim}{x}))\}}{\exp\{\ln(\lambda(\underset{\sim}{x}^*))\}} && \text{(exponential undoes the log)} \\
&= \exp\{\ln(\lambda(\underset{\sim}{x})) - \ln(\lambda(\underset{\sim}{x}^*))\} && \text{(quotient rule for exponentials)} \\
&= \exp\{\beta_j(a-b)\} && \text{(substitute the difference of linear functions)}
\end{aligned}$$

For the means at a common exposure $t > 0$:

$$\begin{aligned}
\frac{\mu(\underset{\sim}{x}, t)}{\mu(\underset{\sim}{x}^*, t)} &= \frac{\lambda(\underset{\sim}{x}) \cdot t}{\lambda(\underset{\sim}{x}^*) \cdot t} && \text{(Poisson regression mean function)} \\
&= \frac{\lambda(\underset{\sim}{x})}{\lambda(\underset{\sim}{x}^*)} && \text{(cancel } t > 0\text{)} \\
&= \lambda(\underset{\sim}{x} : \underset{\sim}{x}^*) && \text{(definition of rate ratio)}
\end{aligned}$$

□

Example 0.1 (Rate ratio for a ten-year age difference)**.**
Suppose a Poisson regression model for emergency-room visits includes age in years, x_1, with coefficient $\beta_1 = 0.03$ and no interactions involving age. By Theorem 0.1 with $a = 60$ and $b = 50$, the rate ratio comparing a 60-year-old to a 50-year-old with the same values of every other covariate is:

$$\begin{aligned} {}_{\lambda}(x_1 = 60 : x_1 = 50) &= \exp\{\beta_1(60 - 50)\} && \text{(rate ratio theorem)} \\ &= \exp\{0.03 \cdot 10\} && \text{(substitute } \beta_1 \text{ and subtract)} \\ &= \exp\{0.3\} && \text{(multiply)} \\ &\approx 1.350 && \text{(evaluate)} \end{aligned}$$

A one-year age difference instead gives $\exp\{0.03\} \approx 1.030$, a 3% higher event rate per year of age. The ten-year rate ratio is the one-year rate ratio multiplied by itself ten times, $\exp\{0.3\} = (\exp\{0.03\})^{10}$, because rate ratios multiply across successive one-year steps.