Sets and Functions

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Remark. Probability and statistics describe events, outcomes, and parameters as sets, and random variables, densities, and measures as functions. This page collects the definitions of sets and functions that the Morrison Lab's probability and statistics notes build on.

Sets

Definition 0.1 (Set)**.** A **set** is a collection of distinct objects, called its elements. We write $a \in A$ when a is an element of the set A, and $a \notin A$ when it is not. Two sets are equal when they have exactly the same elements.

Remark. Definition 0.1 is the informal ("naive") notion of a set, which is all these notes need. Formal set theory states axioms for sets instead (see Wikipedia: Set (mathematics)).

Example 0.1 (Sets of die rolls)**.** The possible results of rolling a six-sided die form the set $\{1, 2, 3, 4, 5, 6\}$, with $3 \in \{1, 2, 3, 4, 5, 6\}$ and $7 \notin \{1, 2, 3, 4, 5, 6\}$. The sets $\{1, 2, 3\}$ and $\{3, 2, 1\}$ are equal, because they have the same elements; the order in which the elements are listed does not matter.

Definition 0.2 (Set-builder notation)**.** For a set A and a statement $P(x)$ about an element x, **set-builder notation** $\{x \in A : P(x)\}$ denotes the set of elements x of A for which $P(x)$ is true.

Example 0.2 (Even die rolls in set-builder notation)**.** $\{x \in \{1, 2, 3, 4, 5, 6\} : x \text{ is even}\} = \{2, 4, 6\}$, and $\{x \in \mathbb{R} : x > 0\}$ is the set of positive real numbers.

Definition 0.3 (Subset)**.** A set A is a **subset** of a set B, written $A \subseteq B$, if every element of A is an element of B.

Example 0.3 (Even die rolls)**.** The even die rolls form a subset of the possible rolls: $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\}$, because each of 2, 4, and 6 is a possible roll. The set $\{2, 4, 7\}$ is not a subset of $\{1, 2, 3, 4, 5, 6\}$, because 7 is not a possible roll.

Theorem 0.1 (Every set is a subset of itself)**.** *For every set A, $A \subseteq A$.*

Proof

Proof. Every element of A is an element of A, which is Definition 0.3 with $B = A$. □

Definition 0.4 (Strict subset)**.** A set A is a **strict subset** of a set B, written $A \subsetneq B$, if A is a subset of B and $A \neq B$.

Remark. Other sources call a strict subset a **proper subset**. Sources disagree about the symbol $\subset$: some use it for "subset" ($\subseteq$), and others for "strict subset" ($\subsetneq$) (see Wikipedia: Subset). These notes avoid $\subset$ and write $\subseteq$ or $\subsetneq$.

Definition 0.5 (Superset)**.** A set B is a **superset** of a set A, written $B \supseteq A$, if A is a subset of B.

Definition 0.6 (Strict superset)**.** A set B is a **strict superset** of a set A, written $B \supsetneq A$, if A is a strict subset of B.

Example 0.4 (Strict subsets and supersets of die rolls)**.** For the even rolls $A = \{2, 4, 6\}$ and the possible rolls $B = \{1, 2, 3, 4, 5, 6\}$:

- $A \subsetneq B$, because $A \subseteq B$ (Example 0.3) and $1 \in B$ but $1 \notin A$, so $A \neq B$;
- $B \supseteq A$ and $B \supsetneq A$, for the same reasons;
- $B \subseteq B$, but B is not a strict subset of itself, because $B = B$.

Definition 0.7 (Empty set)**.** The **empty set**, denoted $\emptyset$, is the set that has no elements.

Remark. Other sources write $\{\}$ (see Wikipedia: Empty set). Some sources call it the **null set**, but in measure theory a "null set" usually means a set of measure zero, which need not be empty.

Example 0.5 (Impossible die rolls)**.** No roll of a six-sided die is greater than 6, so $\{x \in \{1,2,3,4,5,6\} : x > 6\} = \emptyset$.

Theorem 0.2 (There is only one empty set)**.** *If E_1 and E_2 are sets with no elements, then $E_1 = E_2$.*

Proof

Proof. Neither set has any elements, so every element of E_1 is an element of E_2 and vice versa, vacuously. Sets with the same elements are equal (Definition 0.1). □

Theorem 0.3 (The empty set is a subset of every set)**.** *For every set A, $\emptyset \subseteq A$.*

Proof

Proof. By Definition 0.3, $\emptyset \subseteq A$ fails only if some element of $\emptyset$ is not in A. The empty set has no elements, so no such element exists. □

Combining sets

Definition 0.8 (Union)**.** The **union** of sets A and B, written $A \cup B$, is the set of elements that are in A, in B, or in both:

$$A \cup B \stackrel{\text{def}}{=} \{x : x \in A \text{ or } x \in B\}$$

More generally, the union of sets $A_1, A_2, \ldots$, written $\bigcup_i A_i$, is the set of elements that are in at least one A_i.

Definition 0.9 (Intersection)**.** The **intersection** of sets A and B, written $A \cap B$, is the set of elements that are in both A and B:

$$A \cap B \stackrel{\text{def}}{=} \{x : x \in A \text{ and } x \in B\}$$

More generally, the intersection of sets $A_1, A_2, \ldots$, written $\bigcap_i A_i$, is the set of elements that are in every A_i.

Definition 0.10 (Set difference)**.** The **set difference** of sets A and B, written $A \setminus B$, is the set of elements of A that are not in B:

$$A \setminus B \stackrel{\text{def}}{=} \{x \in A : x \notin B\}$$

Example 0.6 (Combining sets of die rolls)**.** Let $A = \{2, 4, 6\}$ (the even rolls) and $B = \{1, 2, 3\}$ (the rolls of at most 3). Then:

- $A \cup B = \{1, 2, 3, 4, 6\}$, the rolls that are even or at most 3;
- $A \cap B = \{2\}$, the only roll that is both;
- $A \setminus B = \{4, 6\}$, the even rolls greater than 3;
- $\{2, 4, 6\} \cap \{1, 3, 5\} = \emptyset$: no roll is both even and odd.

Countable sets

Definition 0.11 (Countable set)**.** A set is **countable** if it is finite, or if its elements can be listed as a sequence $a_1, a_2, a_3, \ldots$ in which every element appears.

Definition 0.12 (Countably infinite set)**.** A set is **countably infinite** if it is countable and not finite.

Example 0.7 (Countable and uncountable sets)**.**

- $\{1, 2, 3, 4, 5, 6\}$ is countable, because it is finite.
- The non-negative integers $\{0, 1, 2, \ldots\}$ are countably infinite: the sequence $0, 1, 2, \ldots$ lists them.
- The integers are countably infinite too: the sequence $0, 1, -1, 2, -2, 3, -3, \ldots$ lists every one of them.

Theorem 0.4 (The unit interval is not countable)**.** *The interval* $[0, 1]$ *is not countable.*

Remark. The standard proof is Cantor's diagonal argument: given any sequence of numbers in $[0, 1]$, it builds a number in $[0, 1]$ whose decimal expansion differs from the nth number's in the nth digit, so the sequence misses it (see Wikipedia: Cantor's diagonal argument).

Functions

Definition 0.13 (Function)**.** A **function** f from a set A to a set B, written $f : A \to B$, assigns to each element $a \in A$ exactly one element $f(a) \in B$, called the value of f at a.

Example 0.8 (Doubling a die roll)**.** The rule $f(a) = 2a$ defines a function $f : \{1, 2, 3, 4, 5, 6\} \to \mathbb{R}$, because it assigns exactly one real number to each possible roll; for example, $f(3) = 6$. A rule that assigned both 2 and -2 to the roll 1 would not be a function, because a function assigns exactly one value to each element.

Definition 0.14 (Domain)**.** The **domain** of a function $f : A \to B$ is the set A of elements that f assigns values to.

Definition 0.15 (Codomain)**.** The **codomain** of a function $f : A \to B$ is the set B that its values are required to lie in.

Definition 0.16 (Image)**.** The **image** of a function $f : A \to B$ is the set of values that f actually takes:

$$f(A) \stackrel{\text{def}}{=} \{f(a) : a \in A\}$$

Remark. Many sources call the image the **range**, but others use "range" for the codomain, so these notes avoid "range" for functions in general (see Wikipedia: Range of a function). The image is always a subset of the codomain, because every value $f(a)$ lies in B.

Example 0.9 (Domain, codomain, and image of the doubled die roll)**.** For the function $f : \{1, 2, 3, 4, 5, 6\} \to \mathbb{R}$ with $f(a) = 2a$ from Example 0.8:

- the domain is $\{1, 2, 3, 4, 5, 6\}$;
- the codomain is $\mathbb{R}$;
- the image is $\{2, 4, 6, 8, 10, 12\}$.

The image is a subset of the codomain, and here a much smaller one: most real numbers, such as 3 and -1, are not values of f.

Extended non-negative real numbers

Definition 0.17 (Extended non-negative real numbers)**.** The **extended non-negative real numbers**, written $[0, \infty]$, are the non-negative real numbers together with an extra element ∞ that is greater than every real number:

$$[0, \infty] \overset{\text{def}}{=} [0, \infty) \cup \{\infty\}$$

Addition extends to $[0, \infty]$ by setting $x + \infty = \infty + x = \infty$ for every $x \in [0, \infty]$.

Remark. ∞ is not a real number, so some real-number arithmetic does not extend to it: $\infty - \infty$ is left undefined, and $\infty + 1 = \infty + 2$ does not imply $1 = 2$. Measure theory uses $[0, \infty]$ because the size of a set, such as the length of the whole real line, can be infinite (see Wikipedia: Extended real number line).

Example 0.10 (Adding with ∞)**.** In $[0, \infty]$, $2 + 3 = 5$ as usual, while $2 + \infty = \infty$ and $\infty + \infty = \infty$.