

Relations

Last modified: 2026-10-27 20:53:54 (PDT)

Remark

Remark. A [relation](#) says which pairs of elements of a [set](#) are related. This page defines the properties that make a relation behave like equality, and the groups of elements that such a relation puts together.

Properties of relations

Definition 0.1 (Reflexive relation). Let $\sim$ be a [relation](#) on a [set](#) S . The relation $\sim$ is **reflexive** [1] if $a \sim a$ for every $a \in S$.

Example 0.1 (Reflexive and non-reflexive relations on the real numbers). On $\mathbb{R}$, the [relation](#) = is [reflexive](#), since $a = a$ for every [real number](#) a . The relation $\leq$ is also reflexive, since $a \leq a$ for every real number a . The relation $<$ is not reflexive, since $2 < 2$ is false.

Definition 0.2 (Symmetric relation). Let $\sim$ be a [relation](#) on a [set](#) S . The relation $\sim$ is **symmetric** [2] if, [for all](#) $a, b \in S$, $a \sim b$ [implies](#) $b \sim a$.

Example 0.2 (Symmetric and non-symmetric relations on the real numbers). On $\mathbb{R}$, the [relation](#) = is [symmetric](#): whenever $a = b$, also $b = a$. The relation $\leq$ is not symmetric, since $2 \leq 5$ is true but $5 \leq 2$ is false.

Equivalence relations

Definition 0.3 (Equivalence relation). Let $\sim$ be a relation on a set S . The relation $\sim$ is an **equivalence relation** [3] if it is **reflexive**, **symmetric**, and **transitive**.

Example 0.3 (Integers with an even difference). On the integers, define the relation $\sim$ by $a \sim b$ when $a - b$ is **even**. For example, $7 \sim 3$, since $7 - 3 = 4 = 2 \cdot 2$, but $7 \not\sim 4$, since $7 - 4 = 3$ is **odd**. The relation $\sim$ is an **equivalence relation**.

Reflexive. For every integer a , $a - a = 0 = 2 \cdot 0$, so $a \sim a$.

Symmetric. Suppose $a \sim b$, so $a - b = 2k$ for some integer k . Then

$$\begin{aligned} b - a &= -(a - b) && \text{(negate the difference)} \\ &= -2k && (a - b = 2k) \\ &= 2(-k), && \text{(factor out 2)} \end{aligned}$$

and $-k$ is an integer, so $b \sim a$.

Transitive. Suppose $a \sim b$ and $b \sim c$, so $a - b = 2j$ and $b - c = 2k$ for some integers j and k . Then

$$\begin{aligned} a - c &= (a - b) + (b - c) && \text{(the } b \text{ terms cancel)} \\ &= 2j + 2k && (a - b = 2j \text{ and } b - c = 2k) \\ &= 2(j + k), && \text{(distributive law)} \end{aligned}$$

and $j + k$ is an integer, so $a \sim c$.

Definition 0.4 (Equivalence class). Let $\sim$ be an **equivalence relation** on a set S , and let $a \in S$. The **equivalence class** [4] of a is the set of elements of S related to a :

$$[a] \stackrel{\text{def}}{=} \{b \in S : b \sim a\}.$$

Example 0.4 (Equivalence classes of integers with an even difference). Let $\sim$ be the **equivalence relation** of Example 0.3 on the integers. The **equivalence class** $[0]$ holds the integers b with $b - 0$ even, which are the **even integers**: $[0] = \{\dots, -4, -2, 0, 2, 4, \dots\}$. The class $[1]$ holds the integers b with $b - 1$ even, which are the **odd integers**: $[1] = \{\dots, -3, -1, 1, 3, 5, \dots\}$. The class $[2]$ is the same **set** as $[0]$, since $2 \sim 0$.

References

- [1] Wikipedia contributors, “Reflexive relation — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Reflexive_relation

- [2] Wikipedia contributors, “Symmetric relation — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Symmetric_relation
- [3] Wikipedia contributors, “Equivalence relation — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Equivalence_relation
- [4] Wikipedia contributors, “Equivalence class — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Equivalence_class