Sigma-Algebras

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σ-algebras

Definition 0.1 (σ-algebra (σ-field, Borel field))**.** A **σ-algebra** [1] (also called a **σ-field** [2] or a **Borel field** [3], Definition 1.2.1) on a set S is a collection $\mathcal{S}$ of subsets of S that satisfies:

- $\mathcal{S}$ contains S itself.
- For each set A in $\mathcal{S}$, $\mathcal{S}$ contains its complement $S \setminus A$.
- For each sequence $A_1, A_2, \ldots$ of sets in $\mathcal{S}$, $\mathcal{S}$ contains their union $\bigcup_{i=1}^{\infty} A_i$.

Remark ("Borel" in "Borel field")

Remark 0.1 ("Borel" in "Borel field"). The "Borel" in "Borel field" is a different use of "Borel" from the Borel σ-algebra on $\mathbb{R}$, which is one particular σ-algebra.

Example 0.1 (σ-algebras for die rolls)**.** For the die rolls $D = \{1, 2, 3, 4, 5, 6\}$ (sets of die rolls), the collection of all subsets of D is a σ-algebra on D, since complements and unions of subsets of D are again subsets of D. So is the smaller collection

$$\{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, D\}$$

which contains D, the complement of each of its sets, and every union of its sets: for example, $\{2, 4, 6\} \cup \{1, 3, 5\} = D$.

Example 0.2 (A collection that is not a σ-algebra)**.** The collection $\{\emptyset, \{1\}, D\}$ is not a σ-algebra on $D = \{1, 2, 3, 4, 5, 6\}$: it contains $\{1\}$ but not its complement $D \setminus \{1\} = \{2, 3, 4, 5, 6\}$.

Example 0.3 (A σ-algebra is closed under complements and countable unions)**.** A σ-algebra $\mathcal{S}$ on a set S is closed under complements in S and under unions of sequences of its sets, because Definition 0.1 requires both. The collection in Example 0.2 is not closed under complements.

Theorem 0.1 (Closure properties of a σ-algebra)**.** *If $\mathcal{S}$ is a σ-algebra on a set S, then $\mathcal{S}$ contains:*

- *the empty set $\emptyset$;*
- *the union $A_1 \cup \cdots \cup A_n$ of any finitely many sets $A_1, \ldots, A_n$ in $\mathcal{S}$;*
- *the intersection $\bigcap_{i=1}^{\infty} A_i$ of any sequence $A_1, A_2, \ldots$ of sets in $\mathcal{S}$.*

So $\mathcal{S}$ is closed under finite unions and under intersections of sequences of its sets.

Proof

Proof. Empty set. $\mathcal{S}$ contains S, so it contains the complement of S, which is $S \setminus S = \emptyset$.
Finite unions. Extend $A_1, \ldots, A_n$ to a sequence by setting

$$\begin{aligned} A_{n+1} &= A_{n+2} \\ &= \cdots \\ &= \emptyset; \end{aligned}$$

each of these sets is in $\mathcal{S}$, by the first part. Then:

$$\begin{aligned} A_1 \cup \cdots \cup A_n &= A_1 \cup \cdots \cup A_n \cup \emptyset \cup \emptyset \cup \cdots && \text{(a union with } \emptyset \text{ adds no elements)} \\ &= \bigcup_{i=1}^{\infty} A_i && (A_i = \emptyset \text{ for } i > n) \end{aligned}$$

and $\mathcal{S}$ contains $\bigcup_{i=1}^{\infty} A_i$ by the union rule of Definition 0.1.
Countable intersections. Each A_i is a subset of S, so an element of S is outside $S \setminus A_i$ exactly when it is in A_i; that is, $S \setminus (S \setminus A_i) = A_i$. So, by De Morgan's laws:

$$\begin{aligned} S \setminus \bigcup_{i=1}^{\infty} (S \setminus A_i) &= \bigcap_{i=1}^{\infty} (S \setminus (S \setminus A_i)) && \text{(De Morgan's laws)} \\ &= \bigcap_{i=1}^{\infty} A_i && (S \setminus (S \setminus A_i) = A_i) \end{aligned}$$

Each $S \setminus A_i$ is in $\mathcal{S}$ by the complement rule, so their union is in $\mathcal{S}$ by the union rule, and the complement of that union is in $\mathcal{S}$ by the complement rule again. □

Theorem 0.2 (All subsets form a σ-algebra)**.** *For any set S, the collection of all subsets of S is a σ-algebra on S.*

Proof

Proof. Each condition of Definition 0.1 holds:

- S is a subset of itself.
- For each subset A of S, the complement $S \setminus A$ contains only elements of S, so it is a subset of S.
- For each sequence $A_1, A_2, \ldots$ of subsets of S, every element of $\bigcup_{i=1}^{\infty} A_i$ is in some A_i and so in S; the union is a subset of S.

□

Theorem 0.3 (An intersection of σ-algebras is a σ-algebra)**.** *Let $\mathcal{S}_j$ be a σ-algebra on a set S for each j in a nonempty set of indices J. Then their intersection $\bigcap_{j \in J} \mathcal{S}_j$, the collection of subsets of S that are in every $\mathcal{S}_j$, is a σ-algebra on S.*

Proof

Proof. Each condition of Definition 0.1 holds:

- S is in every $\mathcal{S}_j$, so it is in the intersection.
- If A is in the intersection, then A is in every $\mathcal{S}_j$, so its complement $S \setminus A$ is in every $\mathcal{S}_j$, and so it is in the intersection.
- If $A_1, A_2, \ldots$ are in the intersection, then they are in every $\mathcal{S}_j$, so their union $\bigcup_{i=1}^{\infty} A_i$ is in every $\mathcal{S}_j$, and so it is in the intersection.

□

Definition 0.2 (σ-algebra generated by a collection)**.** Let $\mathcal{C}$ be a collection of subsets of a set S. The **σ-algebra generated by $\mathcal{C}$**, written $\sigma(\mathcal{C})$, is the intersection of all the σ-algebras on S that contain every set in $\mathcal{C}$.

Remark (The smallest σ-algebra containing $\mathcal{C}$)

Remark 0.2 (The smallest σ-algebra containing $\mathcal{C}$). $\sigma(\mathcal{C})$ is a σ-algebra by Theorem 0.3 (the collection of all subsets of S is one of the σ-algebras intersected, by Theorem 0.2, so there is at least one). It contains every set in $\mathcal{C}$, and it is contained in every other

σ-algebra on S that does, so it is the smallest σ-algebra on S that contains $\mathcal{C}$.

Example 0.4 (The σ-algebra generated by the even rolls)**.** For the die rolls $D = \{1, 2, 3, 4, 5, 6\}$, let $\mathcal{C} = \{\{2, 4, 6\}\}$. Every σ-algebra on D that contains $\{2, 4, 6\}$ must also contain D, the complement $\{1, 3, 5\}$, and $\emptyset$ (Definition 0.1 and Theorem 0.1). The collection $\{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, D\}$ is already a σ-algebra (Example 0.1), so it is $\sigma(\mathcal{C})$.

Definition 0.3 (Borel σ-algebra)**.** The **Borel σ-algebra** on $\mathbb{R}$, written $\mathcal{B}$, is the σ-algebra generated by (Definition 0.2) the collection of all intervals.

Example 0.5 (The Borel σ-algebra contains $\mathbb{R}$ and the half-lines)**.** The interval $(-\infty, 0)$ is in $\mathcal{B}$ by Definition 0.3. So is its complement $[0, \infty)$, because $\mathcal{B}$ is a σ-algebra (Definition 0.1). The set $\mathbb{R}$ is in $\mathcal{B}$ too: a σ-algebra on $\mathbb{R}$ contains $\mathbb{R}$ itself.

Definition 0.4 (Borel set)**.** A **Borel set** [4] is a set in the Borel σ-algebra $\mathcal{B}$.

Example 0.6 (Some Borel sets)**.**

- Every interval, such as $[0, 1]$, $(2, 5)$, or $[0, \infty)$, is a Borel set.
- A single point $\{a\} = [a, a]$ is an interval, so it is a Borel set.
- The integers are a Borel set: they are the union of the sequence of single points $\{0\}, \{1\}, \{-1\}, \{2\}, \{-2\}, \ldots$ (countable and uncountable sets lists them this way), and $\mathcal{B}$ contains that union by Definition 0.1.
- The complement $\mathbb{R} \setminus [0, 1]$ is a Borel set, because $\mathcal{B}$ contains the complement of each of its sets.

[1] Wikipedia contributors, "Sigma-algebra — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/%CF%83-algebra

[2] P. Billingsley, *Probability and measure*, 3rd ed. in Wiley series in probability and mathematical statistics. New York: Wiley, 1995.

[3] G. Casella and R. Berger, *Statistical inference*, 2nd ed. Cengage Learning, 2002. Available: https://www.cengage.com/c/statistical-inference-2e-casella-berger/9780534243128/

[4] Wikipedia contributors, "Borel set — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Borel_set