Product Measures

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Product measures

Definition 0.1 (Measurable rectangle)**.** Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces. A **measurable rectangle** is a Cartesian product $A \times B$ with $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

Definition 0.2 (Product σ-algebra)**.** For the measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$ of Definition 0.1, the **product σ-algebra** $\mathcal{S} \otimes \mathcal{T}$ is the σ-algebra on $S \times T$ generated by the measurable rectangles.

Example 0.1 (Product σ-algebras)**.**

- Give the die rolls D the σ-algebra $\mathcal{S}$ of all its subsets. Each single pair $\{(a, b)\} = \{a\} \times \{b\}$ in $D \times D$ is a measurable rectangle. Every subset of $D \times D$ is a union of finitely many single pairs, and $\mathcal{S} \otimes \mathcal{S}$ contains finite unions of its sets (Theorem 1 in Sigma-Algebras), as well as $\emptyset$, so $\mathcal{S} \otimes \mathcal{S}$ is the collection of all subsets of $D \times D$.
- The rectangle $[0, 2] \times [0, 3]$ is a measurable rectangle for the Borel σ-algebras, so it is in $\mathcal{B} \otimes \mathcal{B}$.

Definition 0.3 (Product measure)**.** Let $(S, \mathcal{S}, \mu)$ and $(T, \mathcal{T}, \nu)$ be measure spaces with σ-finite measures μ and ν. The **product measure** $\mu \otimes \nu$ is the measure on the product σ-algebra $\mathcal{S} \otimes \mathcal{T}$ that gives each measurable rectangle the product of the measures of its sides:

$$(\mu \otimes \nu)(A \times B) = \mu(A)\, \nu(B) \quad \text{for all } A \in \mathcal{S} \text{ and } B \in \mathcal{T},$$

with $0 \cdot \infty = \infty \cdot 0 \stackrel{\text{def}}{=} 0$ and $x \cdot \infty = \infty \cdot x \stackrel{\text{def}}{=} \infty$ for $x > 0$. Exactly one measure has these values [1], Theorem 18.2.

Example 0.2 (Product measures)**.**

- For the counting measure μ on the die rolls D, $(\mu \otimes \mu)(A \times B) = |A||B|$, the number of pairs in $A \times B$. For example, the set of pairs in which both rolls are even is $\{2,4,6\} \times \{2,4,6\}$, with

$$\begin{aligned}(\mu \otimes \mu)(\{2,4,6\} \times \{2,4,6\}) &= 3 \cdot 3 \\ &= 9\end{aligned}$$

  pairs.

- For Lebesgue measure λ on $\mathbb{R}$,

$$\begin{aligned}(\lambda \otimes \lambda)([0,2] \times [0,3]) &= 2 \cdot 3 \\ &= 6,\end{aligned}$$

  the area of the rectangle.

[1] P. Billingsley, *Probability and measure*, 3rd ed. in Wiley series in probability and mathematical statistics. New York: Wiley, 1995.