Measures and Probability Measures

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Measures

Definition 0.1 (Measure)**.** A **measure** on a set S [1] with a σ-algebra $\mathcal{S}$ is a function $\mu : \mathcal{S} \to [0, \infty]$ that satisfies:

- $\mu(\emptyset) = 0$.
- μ is countably additive.

Example 0.1 (Counting elements is a measure)**.** The counting function $\mu(A) \stackrel{\text{def}}{=} |A|$ of Example 1 in Additivity, defined on the σ-algebra of all subsets of D (Example 1 in Sigma-Algebras), takes values in $[0, \infty]$, is countably additive (Example 3 in Additivity), and gives $\mu(\emptyset) = 0$, so it is a measure.

Corollary 0.1 (Measures are finitely additive)**.** *Every measure is finitely additive.*

Proof

Proof. A measure μ is countably additive and has $\mu(\emptyset) = 0$ (Definition 0.1), so Theorem 1 in Additivity applies to it. □

Definition 0.2 (Counting measure)**.** The **counting measure** [2] on a set S is the measure on the σ-algebra of all subsets of S that assigns each finite set its number of elements, and each infinite set the value ∞:

$$\mu(A) \stackrel{\text{def}}{=} \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

Example 0.2 (Counting measure on the non-negative integers)**.** For the counting measure μ on the non-negative integers $\{0, 1, 2, ...\}$, $\mu(\{0, 1, 2\}) = 3$, and the set of even numbers has $\mu(\{0, 2, 4, ...\}) = \infty$. The even numbers are the union of the pairwise disjoint sets $\{0\}, \{2\}, \{4\}, ...$, and countable additivity agrees:

$$\begin{aligned}\sum_{k=0}^{\infty} \mu(\{2k\}) &= 1 + 1 + \cdots \\ &= \infty.\end{aligned}$$

Definition 0.3 (Measurable space)**.** A **measurable space** [3] $(S, \mathcal{S})$ is a set S together with a σ-algebra $\mathcal{S}$ on S.

Definition 0.4 (Measurable set)**.** In a measurable space $(S, \mathcal{S})$, the sets in $\mathcal{S}$ are its **measurable sets**.

Example 0.3 (Measurable spaces of die rolls)**.** For the die rolls $D = \{1, 2, 3, 4, 5, 6\}$:

- $(D, \mathcal{S})$, with $\mathcal{S}$ the collection of all subsets of D, is a measurable space (Example 1 in Sigma-Algebras).
- $(D, \{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, D\})$ is also a measurable space. Its measurable sets are only those four; for example, $\{1\}$ is not one of them.

Definition 0.5 (Measure space)**.** A **measure space** $(S, \mathcal{S}, \mu)$ is a measurable space together with a measure μ on $\mathcal{S}$.

Example 0.4 (Measure space of die rolls)**.** For the die rolls $D = \{1, 2, 3, 4, 5, 6\}$, adding the counting function $\mu(A) = |A|$ of Example 0.1 to the measurable space $(D, \mathcal{S})$ of Example 0.3 makes $(D, \mathcal{S}, \mu)$ a measure space.

Definition 0.6 (Lebesgue measure)**. Lebesgue measure** on $\mathbb{R}$ [4] is the measure λ on the Borel σ-algebra $\mathcal{B}$ that assigns each interval its length: for real numbers $a \leq b$,

$$\begin{aligned}\lambda([a, b]) &= \lambda((a, b)) \\ &= \lambda([a, b)) \\ &= \lambda((a, b]) \\ &\stackrel{\text{def}}{=} b - a,\end{aligned}$$

and $\lambda(I) \stackrel{\text{def}}{=} \infty$ for each unbounded interval I.

Remark (Lebesgue measure exists)

Remark 0.1 (Lebesgue measure exists). Exactly one measure on $\mathcal{B}$ assigns every interval its length; constructing it takes more measure theory than these notes cover [4], which also extends λ to a larger σ-algebra, the Lebesgue-measurable sets, with the same values on Borel sets.

Definition 0.7 (Lebesgue measure on an interval)**. Lebesgue measure on an interval** I, such as $[0, 1]$, is λ (Definition 0.6) applied only to the Borel sets that are subsets of I.

Example 0.5 (Lengths of some Borel sets)**.**

- $$\begin{aligned}\lambda([2, 5]) &= 5 - 2 \\ &= 3.\end{aligned}$$
- A single point has length zero:
$$\begin{aligned}\lambda(\{a\}) &= \lambda([a, a]) \\ &= a - a \\ &= 0.\end{aligned}$$
- The integers are the union of the pairwise disjoint single points $\{0\}, \{1\}, \{-1\}, \{2\}, \{-2\}, \ldots$ (Example 6 in Sigma-Algebras), so countable additivity gives
$$\begin{aligned}\lambda(\mathbb{Z}) &= 0 + 0 + \cdots \\ &= 0.\end{aligned}$$
- $\lambda([0, \infty)) = \infty$, because $[0, \infty)$ is unbounded.
- Lebesgue measure on $[0, 1]$ gives the whole interval $[0, 1]$ the value
$$\begin{aligned}\lambda([0, 1]) &= 1 - 0 \\ &= 1.\end{aligned}$$

Remark (Measures generalize size)

Remark 0.2 (Measures generalize size). A measure generalizes size: it can measure how many elements a set has, as in Definition 0.2, or how long a set of real numbers is, as Lebesgue measure (Definition 0.6) does, assigning each interval $[a, b]$ its length $b - a$. For example, the counting measure on $\{0, 1, 2, ...\}$ gives $\{2, 3, 4, 5\}$ the value 4, while the Lebesgue measure of the interval $[2, 5]$ is $5 - 2 = 3$. Both can serve as the measures μ_1 and μ_2 in the Tonelli and Fubini theorems.

Definition 0.8 (Measure zero, null set)**.** Let $(S, \mathcal{S}, \mu)$ be a measure space. A set $A \in \mathcal{S}$ has **measure zero** [5] (also called a **null set** for μ) if $\mu(A) = 0$.

Example 0.6 (Sets of measure zero)**.**

- For a counting measure, only $\emptyset$ has measure zero: a nonempty set has at least one element, so its counting measure is at least 1.
- For Lebesgue measure, every single point $\{a\}$ and the integers $\mathbb{Z}$ have measure zero (Example 0.5), although neither set is empty.

Definition 0.9 (Almost everywhere (μ-almost everywhere, μ-a.e.))**.** Let $(S, \mathcal{S}, \mu)$ be a measure space. A statement about the elements of S holds **μ-almost everywhere** (written **μ-a.e.**, or **almost everywhere** when μ is clear from context) [6] if the set of elements for which it fails is a subset of a set of measure zero.

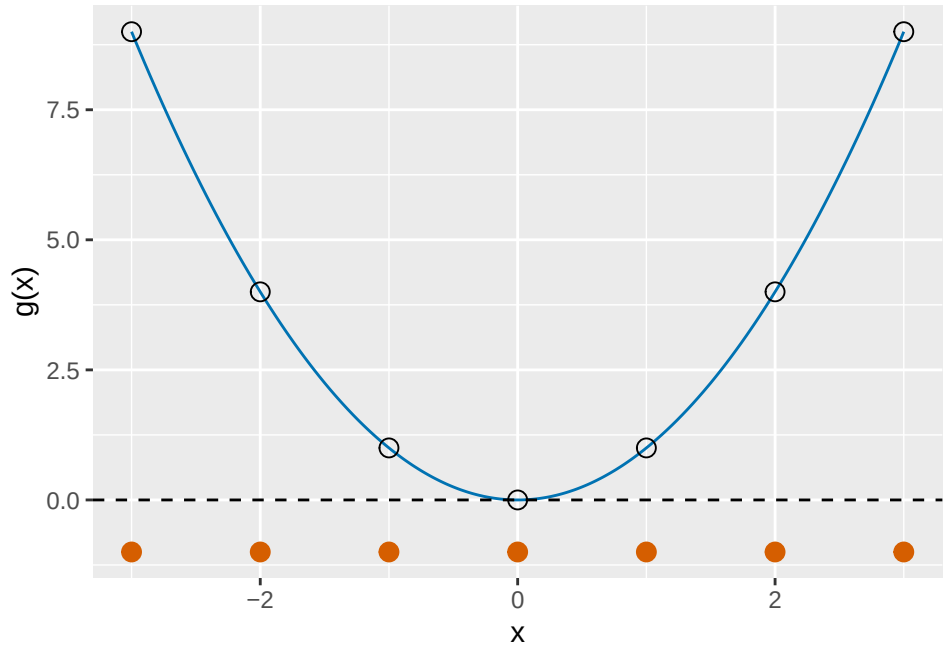
Example 0.7 (Non-negative except on the integers)**.** Let $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = -1$ when x is an integer and $g(x) = x^2$ otherwise. The statement "$g(x) \geq 0$" fails exactly for $x \in \mathbb{Z}$, and $\lambda(\mathbb{Z}) = 0$ (Example 0.5), so $g \geq 0$ holds λ-almost everywhere, though not everywhere.
For a counting measure, only $\emptyset$ has measure zero (Example 0.6), so a statement holds almost everywhere only if it holds everywhere.

Definition 0.10 (Finite measure)**.** Let $(S, \mathcal{S}, \mu)$ be a measure space. The measure μ is **finite** if [7] $\mu(S) < \infty$.

Definition 0.11 (σ-finite measure)**.** Let $(S, \mathcal{S}, \mu)$ be a measure space. The measure μ is **σ-finite** [8] if S is the union $\bigcup_{i=1}^{\infty} A_i$ of a sequence of sets $A_1, A_2, ...$ in $\mathcal{S}$ with $\mu(A_i) < \infty$ for every i.

R

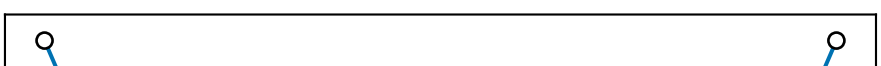
```r
aeg_x_pf <- seq(-3, 3, length.out = 301)
aeg_curve_pf <- tibble::tibble(x = aeg_x_pf, y = aeg_x_pf^2)
aeg_ints_pf <- -3:3
aeg_open_pf <- tibble::tibble(x = aeg_ints_pf, y = aeg_ints_pf^2)
aeg_dots_pf <- tibble::tibble(x = aeg_ints_pf, y = -1)
aeg_aes_pf <- ggplot2::aes(x, y)
ggplot2::ggplot(aeg_curve_pf, aeg_aes_pf) +
  ggplot2::geom_line(colour = "#0072B2") +
  ggplot2::geom_point(data = aeg_open_pf, shape = 1, size = 3) +
  ggplot2::geom_point(data = aeg_dots_pf, colour = "#D55E00", size = 3) +
  ggplot2::geom_hline(yintercept = 0, linetype = "dashed") +
  ggplot2::labs(x = "x", y = "g(x)")
```



Python

```python
import matplotlib.pyplot as plt
import numpy as np

aeg_x_pf = np.linspace(-3, 3, 301)
aeg_ints_pf = np.arange(-3, 4)
fig, ax = plt.subplots(figsize=(5, 3.5), layout="constrained")
ax.plot(aeg_x_pf, aeg_x_pf**2, color="#0072B2")
ax.plot(aeg_ints_pf, aeg_ints_pf**2, linestyle="", marker="o",
        markerfacecolor="white", color="black")
ax.plot(aeg_ints_pf, -np.ones(7), linestyle="", marker="o", color="#D55E00")
ax.axhline(0, linestyle="dashed", color="black")
ax.set_xlabel("x")
ax.set_ylabel("g(x)")
plt.show()
```



Example 0.8 (Finite and σ-finite measures)**.**

- Every finite measure is σ-finite: take $A_1 = S$ and

$$\begin{aligned} A_2 &= A_3 \\ &= \cdots \\ &= \emptyset, \end{aligned}$$

  with $\mu(A_1) = \mu(S) < \infty$ and $\mu(\emptyset) = 0$.

- The counting measure on the die rolls D is finite, with $\mu(D) = 6$.

- The counting measure on $\{0, 1, 2, ...\}$ is not finite, but it is σ-finite: it is the union of the sets $A_i = \{0, 1, \dots, i\}$, each with $\mu(A_i) = i + 1 < \infty$.

- Lebesgue measure on $\mathbb{R}$ is not finite, since $\lambda(\mathbb{R}) = \infty$, but it is σ-finite: $\mathbb{R}$ is the union of the intervals $A_i = [-i, i]$, each with $\lambda(A_i) = 2i < \infty$.

Probability measures

Definition 0.12 (Probability measure)**.** A **probability measure** [9] on a measurable space $(\Omega, \mathcal{F})$ is a measure $\mathbb{P}$ on $\mathcal{F}$ that gives the whole set Ω the value 1:

$$\mathbb{P}(\Omega) = 1$$

Example 0.9 (A fair die, and the unit interval)**.**

- For one roll of a fair die, the sample space is

$$\begin{aligned} \Omega &= D \\ &= \{1, 2, 3, 4, 5, 6\}, \end{aligned}$$

  the events are all subsets of D, and $\mathbb{P}(A) \stackrel{\text{def}}{=} |A|/6$. $\mathbb{P}$ is the counting function $\mu(A) = |A|$ of Example 0.1 divided by 6. So $\mathbb{P}(\emptyset) = 0$, and $\mathbb{P}$ is countably additive, because each sum in Example 3 in Additivity has at most six nonzero terms, and dividing a finite sum by 6 divides each term by 6. It is a probability measure, since

$$\begin{aligned} \mathbb{P}(D) &= 6/6 \\ &= 1. \end{aligned}$$

The outcome 3 is in the event "the roll is odd", $\{1, 3, 5\}$, which has

$$\begin{aligned}\mathbb{P}(\{1, 3, 5\}) &= 3/6 \\ &= 1/2.\end{aligned}$$

- Lebesgue measure on $[0, 1]$ is a probability measure, because $\lambda([0, 1]) = 1$ (Example 0.5).
- The counting measure on D is not a probability measure, because it gives D the value 6.

Definition 0.13 (Probability space)**.** For a probability measure $\mathbb{P}$ (Definition 0.12) on a measurable space $(\Omega, \mathcal{F})$, $(\Omega, \mathcal{F}, \mathbb{P})$ is a **probability space**. [10]

Definition 0.14 (Sample space)**.** In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (Definition 0.13), Ω is its **sample space**. [11]

Definition 0.15 (Outcome)**.** In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (Definition 0.13), the elements of the sample space Ω (Definition 0.14) are **outcomes**. [12]

Definition 0.16 (Event)**.** In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (Definition 0.13), the sets in $\mathcal{F}$ are **events**. [13]

ℹ Remark (Mutually exclusive events)

Remark 0.3 (Mutually exclusive events). In probability, pairwise disjoint events are usually called **mutually exclusive**. For example, for one roll of a six-sided die, the events "the roll is even", $\{2, 4, 6\}$, and "the roll is odd", $\{1, 3, 5\}$, are mutually exclusive: no roll is both even and odd.

[1] Wikipedia contributors, "Measure (mathematics) — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Measure_(mathematics)

[2] Wikipedia contributors, "Counting measure — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Counting_measure

[3] Wikipedia contributors, "Measurable space — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Measurable_space

[4] Wikipedia contributors, "Lebesgue measure — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Lebesgue_measure

[5] Wikipedia contributors, "Null set — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Null_set

[6] Wikipedia contributors, "Almost everywhere — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Almost_everywhere

[7] Wikipedia contributors, "Finite measure — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Finite_measure

[8] Wikipedia contributors, "Sigma-finite measure — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Sigma-finite_measure

[9] Wikipedia contributors, "Probability measure — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Probability_measure

[10] Wikipedia contributors, "Probability space — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Probability_space

[11] Wikipedia contributors, "Sample space — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Sample_space

[12] Wikipedia contributors, "Outcome (probability) — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Outcome_(probability)

[13] Wikipedia contributors, "Event (probability theory) — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Event_(probability_theory)