Measurable Functions and Integrals

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Measurable functions and integrals

Definition 0.1 (Measurable function)**.** Let $(S, \mathcal{S})$ be a measurable space. A function $f : S \to \mathbb{R}$ is **measurable** (with respect to $\mathcal{S}$) [1] if, for every real number c, the set of elements where f is at most c is measurable:

$$\{s \in S : f(s) \leq c\} \in \mathcal{S}$$

Remark (Measurable functions in general)

Remark 0.1 (Measurable functions in general). Other sources define a function $f : S \to T$ between measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$ to be measurable when $\{s \in S : f(s) \in B\} \in \mathcal{S}$ for every $B \in \mathcal{T}$. For $T = \mathbb{R}$ with the Borel σ-algebra, that definition is equivalent to Definition 0.1. For example, with $B = (-\infty, c]$, the general definition requires $\{s \in S : f(s) \leq c\} \in \mathcal{S}$, which is the condition in Definition 0.1.

Example 0.1 (Measurable and not measurable)**.** On the die rolls D, take the σ-algebra $\mathcal{S} = \{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, D\}$ of Example 1 in Sigma-Algebras.

- Let $f(x) = 1$ for even x and $f(x) = 0$ for odd x. The set $\{x \in D : f(x) \leq c\}$ is $\emptyset$ when $c < 0$, $\{1, 3, 5\}$ when $0 \leq c < 1$, and D when $c \geq 1$. All three are in $\mathcal{S}$, so f is measurable.
- Let $g(x) = x$. Then $\{x \in D : g(x) \leq 1\} = \{1\}$, which is not in $\mathcal{S}$, so g is not measurable.

With the σ-algebra of all subsets of D instead, every set $\{x \in D : g(x) \leq c\}$ is a subset of D, so every function $D \to \mathbb{R}$, including g, is measurable.

Definition 0.2 (Integral of a non-negative function)**.** Let $(S, \mathcal{S}, \mu)$ be a measure space, and let $f : S \to [0, \infty)$ be measurable. The **integral** of f with respect to μ is [2]

$$\int_S f \, d\mu \stackrel{\text{def}}{=} \sup \left\{ \sum_{i=1}^{n} \Big(\inf_{s \in A_i} f(s) \Big) \mu(A_i) \right\}$$

where the supremum is over all ways of writing S as the union of finitely many nonempty pairwise disjoint sets $A_1, \dots, A_n$ in $\mathcal{S}$, and each $\inf_{s \in A_i} f(s)$ is the infimum of the values of f on A_i. In the sums, $0 \cdot \infty \stackrel{\text{def}}{=} 0$, and $x \cdot \infty \stackrel{\text{def}}{=} \infty$ for $x > 0$; the supremum is ∞ if some sum is ∞. The integral is also written $\int_S f(s) \, d\mu(s)$.
[3, Sec. 15]

i Remark (Each sum is a lower estimate)

Remark 0.2 (Each sum is a lower estimate). Each sum in Definition 0.2 splits S into pieces and weights each piece A_i by the smallest value f approaches on it, so each sum is at most the integral. For Lebesgue measure on a closed interval $[a, b]$ and a function $f \geq 0$ that has a Riemann integral on $[a, b]$, the two integrals agree: $\int_{[a,b]} f \, d\lambda = \int_a^b f(x) \, dx$ [2].

Example 0.2 (Integrals with respect to counting measure)**.** Let μ be the counting measure on the die rolls D, and let $g : D \to [0, \infty)$. Every function on D is measurable (Example 0.1).
The integral is the sum of the values. For any split of D into nonempty pairwise disjoint sets $A_1, \dots, A_n$:

$$\begin{aligned}
\sum_{i=1}^{n} \Big(\inf_{x \in A_i} g(x) \Big) \mu(A_i) &= \sum_{i=1}^{n} \sum_{x \in A_i} \inf_{y \in A_i} g(y) && (\mu(A_i) \text{ counts the elements of } A_i) \\
&\leq \sum_{i=1}^{n} \sum_{x \in A_i} g(x) && (\text{an infimum is at most each value}) \\
&= \sum_{x \in D} g(x) && (\text{each } x \in D \text{ is in exactly one } A_i)
\end{aligned}$$

The split into the six single points $\{1\}, \dots, \{6\}$ gives exactly $\sum_{x \in D} g(x)$, so the supremum is $\int_D g \, d\mu = \sum_{x \in D} g(x)$. For example, with $g(x) = x$,

$$\begin{aligned}
\int_D g \, d\mu &= 1 + 2 + \cdots + 6 \\
&= 21.
\end{aligned}$$

An infinite integral. For the counting measure μ on $\mathbb{N}$ and the constant function $h(n) = 1$, the split with one set, $A_1 = \mathbb{N}$, gives the sum

$$1 \cdot \mu(\mathbb{N}) = 1 \cdot \infty$$
$$= \infty,$$

so $\int_{\mathbb{N}} h \, d\mu = \infty$.

ℹ Remark (A function in terms of its positive and negative parts)

Remark 0.3 (A function in terms of its positive and negative parts). For a function $f : S \to \mathbb{R}$, with the positive part and negative part, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. (f^+, f^-, and $|f|$ are measurable when f is [3], Theorem 13.3.)

Definition 0.3 (Integral of a real-valued function)**.** Let $(S, \mathcal{S}, \mu)$ be a measure space, and let $f : S \to \mathbb{R}$ be measurable. Let f^+ and f^- be the positive part and negative part of f. The **integral** of f with respect to μ is

$$\int_S f \, d\mu \stackrel{\text{def}}{=} \int_S f^+ \, d\mu - \int_S f^- \, d\mu,$$

using Definition 0.2 for each term, when at least one of the two terms is finite; otherwise, the integral of f is undefined.
[3, Sec. 15]

Example 0.3 (An integral with positive and negative parts)**.** For the counting measure μ on the die rolls D, let $f(x) = x - 3$. Its values at $1, \ldots, 6$ are $-2, -1, 0, 1, 2, 3$, so:

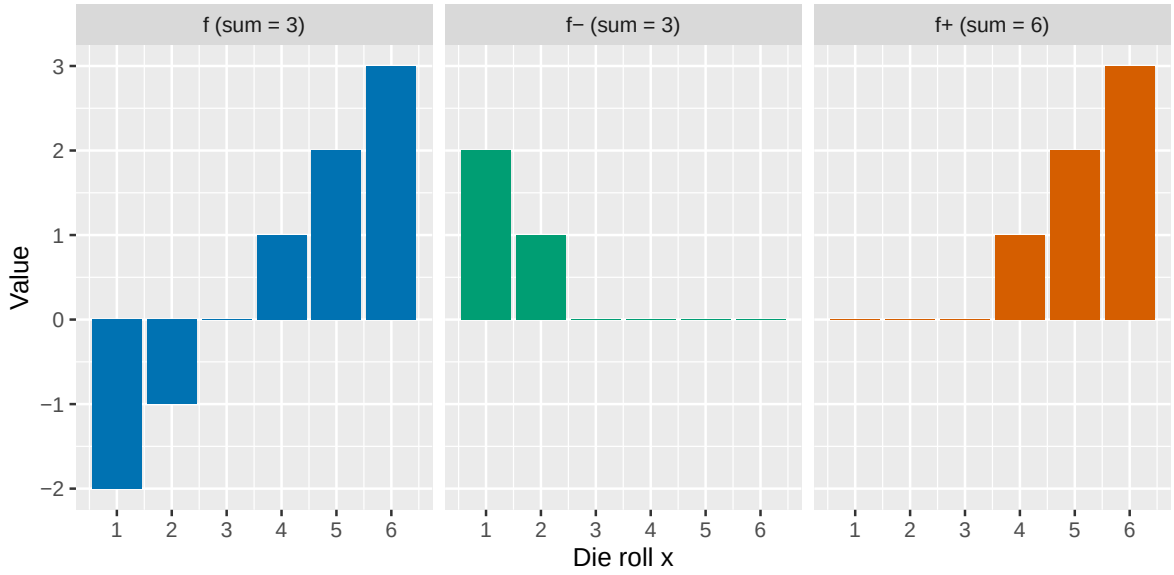
- f^+ has values $0, 0, 0, 1, 2, 3$, and $\int_D f^+ \, d\mu = 6$ by Example 0.2;
- f^- has values $2, 1, 0, 0, 0, 0$, and $\int_D f^- \, d\mu = 3$;
- $$\int_D f \, d\mu = 6 - 3$$
  $$= 3,$$

  which is also

  $$\sum_{x \in D} (x - 3) = 21 - 18$$
  $$= 3.$$

R

```
pnp_rolls_pf <- 1:6
pnp_f_pf <- pnp_rolls_pf - 3
pnp_plus_pf <- pmax(pnp_f_pf, 0)
pnp_minus_pf <- pmax(-pnp_f_pf, 0)
pnp_f_tbl_pf <- tibble::tibble(
  x = pnp_rolls_pf, value = pnp_f_pf, panel = "f (sum = 3)"
)
pnp_plus_tbl_pf <- tibble::tibble(
  x = pnp_rolls_pf, value = pnp_plus_pf, panel = "f+ (sum = 6)"
)
pnp_minus_tbl_pf <- tibble::tibble(
  x = pnp_rolls_pf, value = pnp_minus_pf, panel = "f- (sum = 3)"
)
pnp_bars_pf <- dplyr::bind_rows(pnp_f_tbl_pf, pnp_plus_tbl_pf, pnp_minus_tbl_pf)
pnp_aes_pf <- ggplot2::aes(x, value, fill = panel)
pnp_colours_pf <- c("#0072B2", "#009E73", "#D55E00")
ggplot2::ggplot(pnp_bars_pf, pnp_aes_pf) +
  ggplot2::geom_col() +
  ggplot2::facet_wrap(ggplot2::vars(panel)) +
  ggplot2::scale_fill_manual(values = pnp_colours_pf) +
  ggplot2::scale_x_continuous(breaks = pnp_rolls_pf) +
  ggplot2::guides(fill = "none") +
  ggplot2::labs(x = "Die roll x", y = "Value")
```



Python

```
import matplotlib.pyplot as plt
import numpy as np

pnp_rolls_pf = np.arange(1, 7)
pnp_f_pf = pnp_rolls_pf - 3
pnp_panels_pf = [
    ("f (sum = 3)", pnp_f_pf, "#0072B2"),
    ("f+ (sum = 6)", np.maximum(pnp_f_pf, 0), "#009E73"),
```

Definition 0.4 (Absolutely integrable function, integrable function)**.** Let $(S, \mathcal{S}, \mu)$ be a measure space. A measurable function $f : S \to \mathbb{R}$ is **absolutely integrable** with respect to μ (often just called **integrable**) if the integral of its absolute value is finite:

$$\int_S |f| \, d\mu < \infty$$

The property itself is called absolute integrability.

Remark (An absolutely integrable function has a finite integral)

Remark 0.4 (An absolutely integrable function has a finite integral). If f is absolutely integrable, then $\int_S f^+ \, d\mu$ and $\int_S f^- \, d\mu$ are both finite, because $f^+ \le |f|$ and $f^- \le |f|$, so each sum in Definition 0.2 for f^+ or f^- is at most the matching sum for $|f|$. So $\int_S f \, d\mu$ (Definition 0.3) is a real number.

Example 0.4 (Absolutely integrable and not)**.**

- For the counting measure on the die rolls D, $f(x) = x - 3$ (Example 0.3) is absolutely integrable:

$$\begin{aligned} \int_D |f| \, d\mu &= 2 + 1 + 0 + 1 + 2 + 3 \\ &= 9 \\ &< \infty \end{aligned}$$

  by Example 0.2.

- For the counting measure on $\mathbb{N}$, the constant function $h(n) = 1$ is not absolutely integrable, because

$$\begin{aligned} \int_{\mathbb{N}} |h| \, d\mu &= \int_{\mathbb{N}} h \, d\mu \\ &= \infty \end{aligned}$$

  (Example 0.2).

[1] Wikipedia contributors, "Measurable function — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Measurable_function

[2] Wikipedia contributors, "Lebesgue integration — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Lebesgue_integration

[3] P. Billingsley, *Probability and measure*, 3rd ed. in Wiley series in probability and mathematical statistics. New York: Wiley, 1995.