# Measures

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*Remark.* A measure assigns a size to each set in a collection of sets: how many elements it has, how long it is, or how likely it is. This page builds measures from $\sigma$-algebras and additivity, using the sets and functions page's definitions. The Morrison Lab's probability notes define a probability measure as a measure that gives the whole space size 1 (probability measure).

## $\sigma$-algebras

**Definition 0.1** ($\sigma$-algebra)**.** A **$\sigma$-algebra** on a set $S$ is a collection $\mathcal{S}$ of subsets of $S$ that satisfies:

- $\mathcal{S}$ contains $S$ itself.
- For each set $A$ in $\mathcal{S}$, $\mathcal{S}$ contains its complement $S \setminus A$.
- For each sequence $A_1, A_2, \ldots$ of sets in $\mathcal{S}$, $\mathcal{S}$ contains their union $\bigcup_{i=1}^{\infty} A_i$.

*Remark.* Other sources call it a **$\sigma$-field** (see Wikipedia: $\sigma$-algebra).

**Example 0.1** ($\sigma$-algebras for die rolls)**.** For the die rolls $D = \{1, 2, 3, 4, 5, 6\}$ (sets of die rolls), the collection of all subsets of $D$ is a $\sigma$-algebra on $D$, since complements and unions of subsets of $D$ are again subsets of $D$. So is the smaller collection

$$\{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, D\}$$

which contains $D$, the complement of each of its sets, and every union of its sets: for example, $\{2, 4, 6\} \cup \{1, 3, 5\} = D$.

**Theorem 0.1** (Closure properties of a $\sigma$-algebra)**.** *If $\mathcal{S}$ is a $\sigma$-algebra on a set $S$, then $\mathcal{S}$ contains:*

- *the empty set $\emptyset$;*
- *the union $A_1 \cup \cdots \cup A_n$ of any finitely many sets $A_1, \ldots, A_n$ in $\mathcal{S}$;*
- *the intersection $\bigcap_{i=1}^{\infty} A_i$ of any sequence $A_1, A_2, \ldots$ of sets in $\mathcal{S}$.*

Proof

*Proof. Empty set.* $\mathcal{S}$ contains $S$, so it contains the complement of $S$, which is $S \setminus S = \emptyset$. *Finite unions.* Extend $A_1, \ldots, A_n$ to a sequence by setting $A_{n+1} = A_{n+2} = \cdots = \emptyset$; each of these sets is in $\mathcal{S}$, by the first part. Then:

$$\begin{aligned} A_1 \cup \cdots \cup A_n &= A_1 \cup \cdots \cup A_n \cup \emptyset \cup \emptyset \cup \cdots && \text{(a union with } \emptyset \text{ adds no elements)} \\ &= \bigcup_{i=1}^{\infty} A_i && (A_i = \emptyset \text{ for } i > n) \end{aligned}$$

and $\mathcal{S}$ contains $\bigcup_{i=1}^{\infty} A_i$ by the union rule of Definition 0.1.
*Countable intersections.* An element of $S$ is in every $A_i$ exactly when it is in none of the complements $S \setminus A_i$, so:

$$\bigcap_{i=1}^{\infty} A_i = S \setminus \bigcup_{i=1}^{\infty} (S \setminus A_i) \quad \text{(De Morgan's law)}$$

Each $S \setminus A_i$ is in $\mathcal{S}$ by the complement rule, so their union is in $\mathcal{S}$ by the union rule, and the complement of that union is in $\mathcal{S}$ by the complement rule again. □

**Theorem 0.2** (All subsets form a $\sigma$-algebra)**.** *For any set $S$, the collection of all subsets of $S$ is a $\sigma$-algebra on $S$.*

Proof

*Proof.* Each condition of Definition 0.1 holds:

- $S$ is a subset of itself.
- For each subset $A$ of $S$, the complement $S \setminus A$ contains only elements of $S$, so it is a subset of $S$.
- For each sequence $A_1, A_2, \ldots$ of subsets of $S$, every element of $\bigcup_{i=1}^{\infty} A_i$ is in some $A_i$ and so in $S$; the union is a subset of $S$.

□

### Pairwise disjoint sets

**Definition 0.2** (Pairwise disjoint sets)**.** Finitely or countably many sets $A_1, A_2, \ldots$ are **pairwise disjoint** (also called **disjoint** or **mutually disjoint**) when no two of them share an element:

$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j$$

*Remark.* In probability, pairwise disjoint events are usually called **mutually exclusive**.

**Example 0.2** (Low and high die rolls)**.** For the die rolls $D = \{1,2,3,4,5,6\}$ (sets of die rolls), the sets $\{1,2\}$ and $\{5,6\}$ are pairwise disjoint: no roll is in both. The sets $\{2,4,6\}$ and $\{1,2\}$ are not, because 2 is in both.

### Additivity

**Definition 0.3** (Finite additivity)**.** Let $\mathcal{S}$ be a $\sigma$-algebra on a set $S$. A function $\mu : \mathcal{S} \to [0,\infty]$, with values in the extended non-negative reals, is **finitely additive** if, for every finite collection of pairwise disjoint sets $A_1, \ldots, A_n$ in $\mathcal{S}$, the value of their union is the sum of their values:

$$\mu(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mu(A_i)$$

**Example 0.3** (Counting elements is finitely additive)**.** For the die rolls $D = \{1,2,3,4,5,6\}$ (sets of die rolls), let $\mu(A) \overset{\text{def}}{=} |A|$, the number of elements of $A$, for each set $A$ in the $\sigma$-algebra of all subsets of $D$ (Example 0.1). The sets $\{1,2\}$ and $\{5,6\}$ are pairwise disjoint (Example 0.2), and:

$$\begin{aligned}
\mu(\{1,2\} \cup \{5,6\}) &= \mu(\{1,2,5,6\}) && \text{(take the union)} \\
&= 4 && \text{(count the elements)} \\
&= 2 + 2 && \text{(write 4 as a sum)} \\
&= \mu(\{1,2\}) + \mu(\{5,6\}) && \text{(count each set's elements)}
\end{aligned}$$

The same holds for any pairwise disjoint sets, because the sizes of disjoint sets add, so $\mu$ is finitely additive.

**Definition 0.4** (Countable additivity)**.** Let $\mathcal{S}$ be a $\sigma$-algebra on a set $S$. A function $\mu : \mathcal{S} \to [0, \infty]$ is **countably additive** (also called **$\sigma$-additive**) if, for every sequence of pairwise disjoint sets $A_1, A_2, \ldots$ in $\mathcal{S}$, the value of their union is the sum of their values:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

**Example 0.4** (Counting elements is countably additive)**.** For the counting function $\mu(A) \stackrel{\text{def}}{=} |A|$ of Example 0.3, take any sequence of pairwise disjoint sets $A_1, A_2, \ldots$. No two of them share an element, and $D$ has only six elements, so at most six of the $A_i$ contain any elements; the rest are $\emptyset$, with $\mu(\emptyset) = 0$. The infinite sum therefore has at most six nonzero terms, and finite additivity (Example 0.3) shows that those terms add up to $\mu$ of the union. So $\mu$ is countably additive.

**Lemma 0.1** (Sums of non-negative terms)**.** *Let $a_1, a_2, \ldots$ be values in $[0, \infty]$, with partial sums $s_n \stackrel{def}{=} \sum_{i=1}^{n} a_i$, where $x + \infty = \infty$ for every $x$ in $[0, \infty]$. Then the partial sums are non-decreasing, $s_1 \leq s_2 \leq \cdots$, so their limit, the sum $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n$, always exists, and it is either a finite number or $\infty$.*

Proof

*Proof.* For each $n$:

$$\begin{aligned} s_{n+1} &= s_n + a_{n+1} && \text{(definition of } s_{n+1}\text{)} \\ &\geq s_n && (a_{n+1} \geq 0) \end{aligned}$$

If some partial sum $s_N$ is $\infty$, then $s_n = \infty$ for every $n \geq N$, so the limit is $\infty$. Otherwise, the partial sums form a non-decreasing sequence of real numbers. If that sequence is bounded above, it converges to a finite number (see Wikipedia: Monotone convergence theorem). If it is not bounded above, then for every number $M$ some $s_N$ exceeds $M$, and so does every later $s_n \geq s_N$; the limit is $\infty$. $\square$

*Remark.* By Lemma 0.1, the right-hand side of the equation in Definition 0.4 always has a value.

**Lemma 0.2** (Countable additivity and the empty set)**.** *If $\mu$ is a countably additive function on a $\sigma$-algebra $\mathcal{S}$, then $\mu(\emptyset) = 0$ or $\mu(\emptyset) = \infty$.*

Proof

*Proof.* The set $\emptyset = S \setminus S$ is in $\mathcal{S}$, as the complement of $S$, and the sequence $\emptyset, \emptyset, \ldots$ is pairwise disjoint with union $\emptyset$, so:

$$\begin{aligned} \mu(\emptyset) &= \mu\left(\bigcup_{i=1}^{\infty} \emptyset\right) && \text{(the union of copies of } \emptyset \text{ is } \emptyset) \\ &= \sum_{i=1}^{\infty} \mu(\emptyset) && \text{(countable additivity)} \end{aligned}$$

If $\mu(\emptyset) = c$ for a finite $c > 0$, the right-hand side is $c + c + \cdots = \infty \neq c$, a contradiction. So $\mu(\emptyset)$ is 0 or $\infty$. $\square$

**Example 0.5** (A countably additive function with $\mu(\emptyset) = 0$)**.** The counting function $\mu(A) \stackrel{\text{def}}{=} |A|$ of Example 0.4 is countably additive, and:

$$\begin{aligned} \mu(\emptyset) &= |\emptyset| && \text{(definition of } \mu) \\ &= 0 && (\emptyset \text{ has no elements}) \end{aligned}$$

**Example 0.6** (A countably additive function with $\mu(\emptyset) = \infty$)**.** For the die rolls $D = \{1, 2, 3, 4, 5, 6\}$ (sets of die rolls), let $\mu(A) \stackrel{\text{def}}{=} \infty$ for every set $A$ in the $\sigma$-algebra of all subsets of $D$ (Example 0.1), including $A = \emptyset$. For any sequence of pairwise disjoint sets $A_1, A_2, \ldots$, the left-hand side of the countable additivity equation is:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty \quad \text{(definition of } \mu)$$

and the right-hand side, the limit of its partial sums (Lemma 0.1), is:

$$\begin{aligned} \sum_{i=1}^{\infty} \mu(A_i) &= \infty + \infty + \cdots && \text{(definition of } \mu) \\ &= \infty && \text{(every partial sum is } \infty) \end{aligned}$$

The two sides agree, so $\mu$ is countably additive, and $\mu(\emptyset) = \infty$.

*Remark.* Example 0.5 and Example 0.6 show that both values allowed by Lemma 0.2 occur. So $\mu(\emptyset) = 0$ is an extra requirement, not a consequence of countable additivity.

**Theorem 0.3** (Countable additivity implies finite additivity)**.** *If $\mu$ is a countably additive function on a $\sigma$-algebra $\mathcal{S}$, and $\mu(\emptyset) = 0$, then $\mu$ is finitely additive.*

Proof

*Proof.* Let $A_1, \dots, A_n$ be pairwise disjoint sets in $\mathcal{S}$, and extend them to a sequence by setting $A_{n+1} = A_{n+2} = \dots = \emptyset$. The extended sequence is still pairwise disjoint, since $\emptyset$ shares no element with any set, and its union is $A_1 \cup \dots \cup A_n$. So:

$$\begin{aligned}
\mu(A_1 \cup \dots \cup A_n) &= \mu\left(\bigcup_{i=1}^{\infty} A_i\right) && (A_i = \emptyset \text{ for } i > n) \\
&= \sum_{i=1}^{\infty} \mu(A_i) && (\text{countable additivity}) \\
&= \sum_{i=1}^{n} \mu(A_i) + \sum_{i=n+1}^{\infty} \mu(\emptyset) && (A_i = \emptyset \text{ for } i > n) \\
&= \sum_{i=1}^{n} \mu(A_i) && (\mu(\emptyset) = 0)
\end{aligned}$$

□

**Example 0.7** (Finitely additive but not countably additive)**.** Let $S = \{0, 1, 2, \dots\}$, with the $\sigma$-algebra of all subsets of $S$ (Theorem 0.2), and define:

$$\mu(A) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

*$\mu$ is finitely additive.* Let $A_1, \dots, A_n$ be pairwise disjoint subsets of $S$. If every $A_i$ is finite, then so is their union, and:

$$\begin{aligned}
\mu(A_1 \cup \dots \cup A_n) &= 0 && (\text{a finite union of finite sets is finite}) \\
&= \sum_{i=1}^{n} 0 && (\text{a sum of zeros is } 0) \\
&= \sum_{i=1}^{n} \mu(A_i) && (\text{each } A_i \text{ is finite})
\end{aligned}$$

If some $A_k$ is infinite, then the union, which contains $A_k$, is infinite too, and:

$$\begin{aligned}
\mu(A_1 \cup \dots \cup A_n) &= \infty && (\text{the union is infinite}) \\
&= \mu(A_k) + \sum_{i \neq k} \mu(A_i) && (\mu(A_k) = \infty, \text{ and } \infty + x = \infty) \\
&= \sum_{i=1}^{n} \mu(A_i) && (\text{regroup the terms})
\end{aligned}$$

*$\mu$ is not countably additive.* The single-element sets $\{0\}, \{1\}, \{2\}, \dots$ are pairwise disjoint, and their union is $S$, which is infinite. So:

$$\begin{aligned} \mu\left(\bigcup_{k=0}^{\infty}\{k\}\right) &= \mu(S) && \text{(the union is } S\text{)} \\ &= \infty && (S \text{ is infinite}) \end{aligned}$$

but:

$$\begin{aligned} \sum_{k=0}^{\infty}\mu(\{k\}) &= 0+0+\cdots && \text{(each } \{k\} \text{ is finite)} \\ &= 0 && \text{(every partial sum is 0)} \end{aligned}$$

*Remark.* Example 0.7 shows that the converse of Theorem 0.3 fails, even when $\mu(\emptyset) = 0$: a finitely additive function need not be countably additive (see Wikipedia: Sigma-additive set function — An additive function which is not $\sigma$-additive).

## Measures

**Definition 0.5** (Measure)**.** A **measure** on a set $S$ with a $\sigma$-algebra $\mathcal{S}$ is a function $\mu : \mathcal{S} \to [0, \infty]$ that satisfies:

- $\mu(\emptyset) = 0$.
- $\mu$ is countably additive.

**Example 0.8** (Counting elements is a measure)**.** The counting function $\mu(A) \overset{\text{def}}{=} |A|$ of Example 0.3, defined on the $\sigma$-algebra of all subsets of $D$ (Example 0.1), takes values in $[0, \infty]$, is countably additive (Example 0.4), and gives $\mu(\emptyset) = 0$, so it is a measure.

**Corollary 0.1** (Measures are finitely additive)**.** *Every measure is finitely additive.*

Proof

*Proof.* A measure $\mu$ is countably additive and has $\mu(\emptyset) = 0$ (Definition 0.5), so Theorem 0.3 applies to it. □

**Definition 0.6** (Counting measure)**.** The **counting measure** on a set $S$ is the measure on the $\sigma$-algebra of all subsets of $S$ that assigns each finite set its number of elements, and each infinite set the value $\infty$:

$$\mu(A) \stackrel{\text{def}}{=} \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

**Example 0.9** (Counting measure on the non-negative integers)**.** For the counting measure $\mu$ on $\{0, 1, 2, \ldots\}$, $\mu(\{0, 1, 2\}) = 3$, and the set of even numbers has $\mu(\{0, 2, 4, \ldots\}) = \infty$. The even numbers are the union of the pairwise disjoint sets $\{0\}, \{2\}, \{4\}, \ldots$, and countable additivity agrees: $\sum_{k=0}^{\infty} \mu(\{2k\}) = 1 + 1 + \cdots = \infty$.

*Remark.* A measure generalizes size: it can measure how many elements a set has, as in Definition 0.6, or how long a set of real numbers is, as Lebesgue measure does, assigning each interval $[a, b]$ its length $b - a$. Both appear as reference measures in the Fubini–Tonelli theorem.