Additivity

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Additivity

Definition 0.1 (Finite additivity)**.** Let $\mathcal{S}$ be a σ-algebra on a set S. A function $\mu : \mathcal{S} \to [0, \infty]$, with values in the extended non-negative reals, is **finitely additive** [1] if, for every finite collection of pairwise disjoint sets $A_1, \ldots, A_n$ in $\mathcal{S}$, the value of their union is the sum of their values:

$$\mu(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mu(A_i)$$

Example 0.1 (Counting elements is finitely additive)**.** For the die rolls $D = \{1, 2, 3, 4, 5, 6\}$ (sets of die rolls), let $\mu(A) \stackrel{\text{def}}{=} |A|$, the number of elements of A (cardinality), for each set A in the σ-algebra of all subsets of D (Example 1 in Sigma-Algebras). The sets $\{1, 2\}$ and $\{5, 6\}$ are pairwise disjoint (low, middle and high die rolls), and:

$$\begin{aligned}
\mu(\{1,2\} \cup \{5,6\}) &= \mu(\{1,2,5,6\}) && \text{(take the union)} \\
&= 4 && \text{(count the elements)} \\
&= 2 + 2 && \text{(write 4 as a sum)} \\
&= \mu(\{1,2\}) + \mu(\{5,6\}) && \text{(count each set's elements)}
\end{aligned}$$

The same holds for any pairwise disjoint sets, because the sizes of disjoint sets add, so μ is finitely additive.

Example 0.2 (Squaring the count is not finitely additive)**.** On the same σ-algebra, let $\nu(A) \stackrel{\text{def}}{=} |A|^2$. The sets $\{1\}$ and $\{2\}$ are pairwise disjoint, but

$$\begin{aligned}
\nu(\{1\} \cup \{2\}) &= \nu(\{1,2\}) && \text{(take the union)} \\
&= 2^2 \\
&= 4, && \text{(count, then square)} \\
\nu(\{1\}) + \nu(\{2\}) &= 1^2 + 1^2 \\
&= 2, && \text{(count each set, then square)}
\end{aligned}$$

and $4 \neq 2$, so ν is not finitely additive.

Definition 0.2 (Countable additivity (σ-additivity))**.** Let $\mathcal{S}$ be a σ-algebra on a set S. A function $\mu : \mathcal{S} \to [0, \infty]$ is **countably additive** [1] (also called **σ-additive**) if, for every sequence of pairwise disjoint sets $A_1, A_2, \ldots$ in $\mathcal{S}$, the value of their union is the sum of their values:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Example 0.3 (Counting elements is countably additive)**.** For the counting function $\mu(A) \stackrel{\text{def}}{=} |A|$ of Example 0.1, take any sequence of pairwise disjoint sets $A_1, A_2, \ldots$. No two of them share an element, and D has only six elements, so at most six of the A_i contain any elements; the rest are $\emptyset$, with $\mu(\emptyset) = 0$. The infinite sum therefore has at most six nonzero terms, and finite additivity (Example 0.1) shows that those terms add up to μ of the union. So μ is countably additive.

Remark (The sum in countable additivity always has a value)

Remark 0.1 (The sum in countable additivity always has a value). By sums of non-negative terms, the right-hand side of the equation in Definition 0.2, $\sum_{i=1}^{\infty} \mu(A_i)$, always has a value, because each $\mu(A_i)$ is in $[0, \infty]$. For example, if $\mu(A_i) = 1/2^i$ for each i, the partial sums are $s_n = 1 - 1/2^n$, and the sum is 1; if $\mu(A_i) = 1$ for each i, the partial sums are $s_n = n$, and the sum is ∞.

Lemma 0.1 (Countable additivity and the empty set)**.** *If μ is a countably additive function on a σ-algebra $\mathcal{S}$, then $\mu(\emptyset) = 0$ or $\mu(\emptyset) = \infty$.*

Proof

Proof. The set $\emptyset = S \setminus S$ is in $\mathcal{S}$, as the complement of S, and the sequence $\emptyset, \emptyset, \ldots$ is pairwise disjoint with union $\emptyset$, so:

$$\begin{aligned}\mu(\emptyset) &= \mu\left(\bigcup_{i=1}^{\infty} \emptyset\right) && \text{(the union of copies of } \emptyset \text{ is } \emptyset\text{)} \\ &= \sum_{i=1}^{\infty} \mu(\emptyset) && \text{(countable additivity)}\end{aligned}$$

If $\mu(\emptyset) = c$ for a finite $c > 0$, the right-hand side is $c + c + \cdots = \infty \neq c$, a contradiction. So $\mu(\emptyset)$ is 0 or ∞. □

Example 0.4 (A countably additive function with $\mu(\emptyset) = 0$)**.** The counting function $\mu(A) \stackrel{\text{def}}{=} |A|$ of Example 0.3 is countably additive, and:

$$\begin{aligned}\mu(\emptyset) &= |\emptyset| && \text{(definition of } \mu\text{)} \\ &= 0 && (\emptyset \text{ has no elements})\end{aligned}$$

Example 0.5 (A countably additive function with $\mu(\emptyset) = \infty$)**.** For the die rolls $D = \{1, 2, 3, 4, 5, 6\}$ (sets of die rolls), let $\mu(A) \stackrel{\text{def}}{=} \infty$ for every set A in the σ-algebra of all subsets of D (Example 1 in Sigma-Algebras), including $A = \emptyset$. For any sequence of pairwise disjoint sets $A_1, A_2, ...$, the left-hand side of the countable additivity equation is:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty \quad \text{(definition of } \mu\text{)}$$

and the right-hand side, the limit of its partial sums (sums of non-negative terms), is:

$$\begin{aligned}\sum_{i=1}^{\infty} \mu(A_i) &= \infty + \infty + \cdots && \text{(definition of } \mu\text{)} \\ &= \infty && \text{(every partial sum is } \infty\text{)}\end{aligned}$$

The two sides agree, so μ is countably additive, and $\mu(\emptyset) = \infty$.

ℹ Remark ($\mu(\emptyset) = 0$ is an extra requirement)

Remark 0.2 ($\mu(\emptyset) = 0$ is an extra requirement). Example 0.4 and Example 0.5 show that both values allowed by Lemma 0.1 occur for countably additive functions. So $\mu(\emptyset) = 0$ is an extra requirement, not a consequence of countable additivity.

Theorem 0.1 (Countable additivity implies finite additivity)**.** *If μ is a countably additive function on a σ-algebra $\mathcal{S}$, and $\mu(\emptyset) = 0$, then μ is finitely additive.*

Proof

Proof. Let $A_1, \dots, A_n$ be pairwise disjoint sets in $\mathcal{S}$, and extend them to a sequence by setting

$$\begin{aligned} A_{n+1} &= A_{n+2} \\ &= \cdots \\ &= \emptyset. \end{aligned}$$

The extended sequence is still pairwise disjoint, since $\emptyset$ shares no element with any set, and its union is $A_1 \cup \cdots \cup A_n$. So:

$$\begin{aligned} \mu(A_1 \cup \cdots \cup A_n) &= \mu\left(\bigcup_{i=1}^{\infty} A_i\right) && (A_i = \emptyset \text{ for } i > n) \\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{(countable additivity)} \\ &= \sum_{i=1}^{n} \mu(A_i) + \sum_{i=n+1}^{\infty} \mu(\emptyset) && (A_i = \emptyset \text{ for } i > n) \\ &= \sum_{i=1}^{n} \mu(A_i) && (\mu(\emptyset) = 0) \end{aligned}$$

□

Example 0.6 (Finitely additive but not countably additive)**.** Let $S = \{0, 1, 2, ...\}$, with the σ-algebra of all subsets of S (Theorem 2 in Sigma-Algebras), and define:

$$\mu(A) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

μ is finitely additive. Let $A_1, \dots, A_n$ be pairwise disjoint subsets of S. If every A_i is finite, then so is their union, and:

$$\begin{aligned} \mu(A_1 \cup \cdots \cup A_n) &= 0 && \text{(a finite union of finite sets is finite)} \\ &= \sum_{i=1}^{n} 0 && \text{(a sum of zeros is 0)} \\ &= \sum_{i=1}^{n} \mu(A_i) && \text{(each } A_i \text{ is finite)} \end{aligned}$$

If some A_k is infinite, then the union, which contains A_k, is infinite too, and:

$$\begin{aligned}
\mu(A_1 \cup \cdots \cup A_n) &= \infty && \text{(the union is infinite)} \\
&= \mu(A_k) + \sum_{i \in \{1,\ldots,n\} \setminus \{k\}} \mu(A_i) && (\mu(A_k) = \infty, \text{ and } \infty + x = \infty) \\
&= \sum_{i=1}^{n} \mu(A_i) && \text{(regroup the terms)}
\end{aligned}$$

μ is not countably additive. The single-element sets $\{0\}, \{1\}, \{2\}, \ldots$ are pairwise disjoint, and their union is S, which is infinite. So:

$$\begin{aligned}
\mu\left(\bigcup_{k=0}^{\infty} \{k\}\right) &= \mu(S) && \text{(the union is } S\text{)} \\
&= \infty && (S \text{ is infinite})
\end{aligned}$$

but:

$$\begin{aligned}
\sum_{k=0}^{\infty} \mu(\{k\}) &= 0 + 0 + \cdots && \text{(each } \{k\} \text{ is finite)} \\
&= 0 && \text{(every partial sum is 0)}
\end{aligned}$$

Remark (Finite additivity does not imply countable additivity)

Remark 0.3 (Finite additivity does not imply countable additivity). Example 0.6 shows that the converse of Theorem 0.1 fails, even when $\mu(\emptyset) = 0$: a finitely additive function need not be countably additive [1]. The function μ in Example 0.6 has $\mu(\emptyset) = 0$, because $\emptyset$ is finite.

[1] Wikipedia contributors, "Sigma-additive set function — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Sigma-additive_set_function