# Orthogonal Complements and the Fundamental Theorem

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## Orthogonal complements

**Theorem 0.1** (The dot product is linear in each slot)**.** *For vectors* $\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{w} \in \mathbb{R}^p$ *and numbers* $a, b$*,*

$$\underset{\sim}{x} \cdot (a\,\underset{\sim}{u} + b\,\underset{\sim}{w}) = a\,(\underset{\sim}{x} \cdot \underset{\sim}{u}) + b\,(\underset{\sim}{x} \cdot \underset{\sim}{w})$$

*and*

$$(a\,\underset{\sim}{u} + b\,\underset{\sim}{w}) \cdot \underset{\sim}{x} = a\,(\underset{\sim}{u} \cdot \underset{\sim}{x}) + b\,(\underset{\sim}{w} \cdot \underset{\sim}{x}).$$

Proof

*Proof.* **Second slot.**

$$
\begin{aligned}
\underset{\sim}{x} \cdot (a\,\underset{\sim}{u} + b\,\underset{\sim}{w}) &= \sum_{i=1}^{p} x_i\,(a\,\underset{\sim}{u} + b\,\underset{\sim}{w})_i && \text{(Definition 10 in Vectors)} \\
&= \sum_{i=1}^{p} x_i\,(a u_i + b w_i) && \text{(Definition 9 in Matrices, Definition 9 in Vectors)} \\
&= \sum_{i=1}^{p} (x_i a u_i + x_i b w_i) && \text{(distribute each } x_i\text{)} \\
&= \sum_{i=1}^{p} (a\,x_i u_i + b\,x_i w_i) && \text{(commute the factors in each product)} \\
&= \sum_{i=1}^{p} a\,x_i u_i + \sum_{i=1}^{p} b\,x_i w_i && \text{(split the finite sum)} \\
&= a \sum_{i=1}^{p} x_i u_i + b \sum_{i=1}^{p} x_i w_i && \text{(factor } a \text{ and } b \text{ out of the sums)} \\
&= a\,(\underset{\sim}{x} \cdot \underset{\sim}{u}) + b\,(\underset{\sim}{x} \cdot \underset{\sim}{w}). && \text{(Definition 10 in Vectors)}
\end{aligned}
$$

**First slot.**

$$
\begin{aligned}
(a\,\underset{\sim}{u} + b\,\underset{\sim}{w}) \cdot \underset{\sim}{x} &= \underset{\sim}{x} \cdot (a\,\underset{\sim}{u} + b\,\underset{\sim}{w}) && \text{(Theorem 1 in Vectors)} \\
&= a\,(\underset{\sim}{x} \cdot \underset{\sim}{u}) + b\,(\underset{\sim}{x} \cdot \underset{\sim}{w}) && \text{(second slot)} \\
&= a\,(\underset{\sim}{u} \cdot \underset{\sim}{x}) + b\,(\underset{\sim}{w} \cdot \underset{\sim}{x}). && \text{(Theorem 1 in Vectors, twice)}
\end{aligned}
$$

Two special cases are used often. With

$$
\begin{aligned}
a &= b \\
&= 1,
\end{aligned}
$$

and $1\,\underset{\sim}{u} = \underset{\sim}{u}$, the theorem gives $\underset{\sim}{x} \cdot (\underset{\sim}{u} + \underset{\sim}{w}) = \underset{\sim}{x} \cdot \underset{\sim}{u} + \underset{\sim}{x} \cdot \underset{\sim}{w}$. With $\underset{\sim}{w} = \underset{\sim}{u}$ and $b = 0$, and $0\,\underset{\sim}{u} = \underset{\sim}{0}$, it gives $\underset{\sim}{x} \cdot (a\,\underset{\sim}{u}) = a\,(\underset{\sim}{x} \cdot \underset{\sim}{u})$, and likewise in the first slot. □

**Example 0.1** (Splitting a dot product)**.** Let $\underset{\sim}{x} = (1, 2)$, $\underset{\sim}{u} = (3, 0)$, $\underset{\sim}{w} = (0, 1)$, $a = 2$ and $b = -1$. Directly, $a\,\underset{\sim}{u} + b\,\underset{\sim}{w} = (6, -1)$ and

$$
\begin{aligned}
\underset{\sim}{x} \cdot (6, -1) &= 6 - 2 \\
&= 4.
\end{aligned}
$$

By Theorem 0.1,

$$
\begin{aligned}
a\,(\underset{\sim}{x} \cdot \underset{\sim}{u}) + b\,(\underset{\sim}{x} \cdot \underset{\sim}{w}) &= 2 \cdot 3 + (-1) \cdot 2 \\
&= 4,
\end{aligned}
$$

the same number.

**Exercise 0.1** (Finding a dot product from two known dot products)**.** Theorem 0.1 lets a dot product with a combination be found from the dot products with its parts. Suppose that $\underset{\sim}{x} \cdot \underset{\sim}{u} = 3$ and $\underset{\sim}{x} \cdot \underset{\sim}{w} = -1$.
Find $\underset{\sim}{x} \cdot (2\,\underset{\sim}{u} + 5\,\underset{\sim}{w})$.

**Hint**

Use Theorem 0.1 with $a = 2$ and $b = 5$.

Solution

*Solution* 0.1. By Theorem 0.1,

$$\underset{\sim}{x} \cdot (2\,\underset{\sim}{u} + 5\,\underset{\sim}{w}) = 2\,(\underset{\sim}{x} \cdot \underset{\sim}{u}) + 5\,(\underset{\sim}{x} \cdot \underset{\sim}{w}) = 2 \cdot 3 + 5 \cdot (-1) = 1.$$

**Definition 0.1** (Orthogonal complement)**.** The **orthogonal complement** [1] of a set $\mathcal{X}$ of vectors in $\mathbb{R}^p$ is the set of vectors orthogonal (Definition 22 in Vectors) to every vector of $\mathcal{X}$:

$$\mathcal{X}^{\perp} \stackrel{\text{def}}{=} \{\underset{\sim}{u} \in \mathbb{R}^p : \underset{\sim}{x} \cdot \underset{\sim}{u} = 0 \text{ for all } \underset{\sim}{x} \in \mathcal{X}\}.$$

$\mathcal{X}^{\perp}$ is read "$\mathcal{X}$ perp".

**Example 0.2** (Orthogonal complements in $\mathbb{R}^3$)**.**

- **A plane.** Let $\mathcal{X} = \{(a, b, 0) : a, b \in \mathbb{R}\}$, the plane $z = 0$. A vector $\underset{\sim}{u}$ is in $\mathcal{X}^{\perp}$ when

$$\begin{aligned}(a, b, 0) \cdot \underset{\sim}{u} &= au_1 + bu_2 \\ &= 0\end{aligned}$$

  for all $a$ and $b$. Taking $(a, b) = (1, 0)$ forces $u_1 = 0$, and taking $(a, b) = (0, 1)$ forces $u_2 = 0$; conversely, if

$$\begin{aligned}u_1 &= u_2 \\ &= 0,\end{aligned}$$

  then $au_1 + bu_2 = 0$ for all $a$ and $b$. So $\mathcal{X}^{\perp} = \{(0, 0, c) : c \in \mathbb{R}\}$, the $z$-axis.

- **Two vectors.** The two-element set $\{(1, 0, 0), (0, 1, 0)\}$ is not a subspace, and its orthogonal complement is the $z$-axis too: the two conditions $u_1 = 0$ and $u_2 = 0$ are the same as for the plane.

- **The extremes.** $\{\underset{\sim}{0}\}^\perp = \mathbb{R}^3$, because $\underset{\sim}{0} \cdot \underset{\sim}{u} = 0$ for every $\underset{\sim}{u}$. $(\mathbb{R}^3)^\perp = \{\underset{\sim}{0}\}$, because a $\underset{\sim}{u}$ in it is orthogonal to itself, and

$$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{u} &= u_1^2 + u_2^2 + u_3^2 \\ &= 0 \end{aligned}$$

forces $\underset{\sim}{u} = \underset{\sim}{0}$.

**Exercise 0.2** (Testing whether a vector is in an orthogonal complement)**.** A vector is in the orthogonal complement of a set when it is orthogonal to every vector of the set. Let $\mathcal{X} = \{(1, 2, 0), (0, 3, 1)\}$, a set of two vectors in $\mathbb{R}^3$.
Decide whether the vector $(2, -1, 3)$ is in $\mathcal{X}^\perp$.

**Hint**

Take the dot product of $(2, -1, 3)$ with each vector of $\mathcal{X}$.

Solution

*Solution* 0.2. Here

$$\begin{aligned} (1, 2, 0) \cdot (2, -1, 3) &= 2 - 2 + 0 = 0, \\ (0, 3, 1) \cdot (2, -1, 3) &= 0 - 3 + 3 = 0, \end{aligned}$$

so $(2, -1, 3)$ is orthogonal to both vectors of $\mathcal{X}$, and it is in $\mathcal{X}^\perp$.

Remark (The orthogonal complement of a set that is not a subspace)

*Remark* 0.1 (The orthogonal complement of a set that is not a subspace). $\mathcal{X}$ in Definition 0.1 need not be a subspace.

**Theorem 0.2** (An orthogonal complement is a subspace)**.** *For any set $\mathcal{X}$ of vectors in $\mathbb{R}^p$, $\mathcal{X}^\perp$ (Definition 0.1) is a subspace of $\mathbb{R}^p$ (Definition 4 in Subspaces and Rank).*

Proof

*Proof.* $\mathcal{X}^\perp$ is not empty:

$$\begin{aligned} \underset{\sim}{x} \cdot \underset{\sim}{0} &= \sum_i x_i \cdot 0 \\ &= 0 \end{aligned}$$

for every $\underset{\sim}{x}$, so $\underset{\sim}{0} \in \mathcal{X}^\perp$. Let $\underset{\sim}{u}, \underset{\sim}{w} \in \mathcal{X}^\perp$, let $c \in \mathbb{R}$, and let $\underset{\sim}{x} \in \mathcal{X}$.
**Addition.**

$$\begin{aligned} \underset{\sim}{x} \cdot (\underset{\sim}{u} + \underset{\sim}{w}) &= \underset{\sim}{x} \cdot \underset{\sim}{u} + \underset{\sim}{x} \cdot \underset{\sim}{w} && \text{(Theorem 0.1, special case } a \text{ and } b \text{ both equal to 1)} \\ &= 0 + 0 && (\underset{\sim}{u}, \underset{\sim}{w} \in \mathcal{X}^\perp) \\ &= 0. && \text{(arithmetic)} \end{aligned}$$

**Scalar multiplication.**

$$\begin{aligned} \underset{\sim}{x} \cdot (c\,\underset{\sim}{u}) &= c\,(\underset{\sim}{x} \cdot \underset{\sim}{u}) && \text{(Theorem 0.1, special case } b = 0) \\ &= c \cdot 0 && (\underset{\sim}{u} \in \mathcal{X}^\perp) \\ &= 0. && \text{(arithmetic)} \end{aligned}$$

These two equations hold for every $\underset{\sim}{x} \in \mathcal{X}$, so $\underset{\sim}{u} + \underset{\sim}{w}$ and $c\,\underset{\sim}{u}$ are in $\mathcal{X}^\perp$. □

**Example 0.3** (The complement of a single vector is a line)**.** The one-element set $\mathcal{X} = \{(1, 2)\}$ in $\mathbb{R}^2$ is not a subspace: $2\,(1, 2) = (2, 4)$ is not in it. Its orthogonal complement is

$$\begin{aligned} \mathcal{X}^\perp &= \{\underset{\sim}{u} \in \mathbb{R}^2 : u_1 + 2u_2 = 0\} \\ &= \{c\,(-2, 1) : c \in \mathbb{R}\}, \end{aligned}$$

since $u_1 + 2u_2 = 0$ means

$$\begin{aligned} \underset{\sim}{u} &= (-2u_2, u_2) \\ &= u_2\,(-2, 1). \end{aligned}$$

That set is span$\{(-2, 1)\}$, a subspace by Theorem 2 in Subspaces and Rank, as Theorem 0.2 says it must be. It contains $(-2, 1)$, the vector Remark 8 in Vectors found perpendicular to $(1, 2)$.

**Exercise 0.3** (Finding the orthogonal complement of two vectors)**.** Theorem 0.2 says that an orthogonal complement is always a subspace, even when the original set is not. Let $\mathcal{X} = \{(1, 0, 1), (1, 0, -1)\}$, a set of two vectors in $\mathbb{R}^3$.
Find $\mathcal{X}^\perp$ as a span.

**Hint**

A vector $\underset{\sim}{u}$ is in $\mathcal{X}^\perp$ when $u_1 + u_3 = 0$ and $u_1 - u_3 = 0$.

Solution

*Solution* 0.3. A vector $\underset{\sim}{u}$ is in $\mathcal{X}^\perp$ exactly when $(1,0,1)\cdot\underset{\sim}{u} = u_1+u_3 = 0$ and $(1,0,-1)\cdot\underset{\sim}{u} = u_1 - u_3 = 0$. Adding the two equations gives $u_1 = 0$, and then $u_3 = 0$, while $u_2$ is free. So

$$\mathcal{X}^\perp = \{(0,u_2,0) : u_2 \in \mathbb{R}\} = \text{span}\{(0,1,0)\},$$

the second coordinate axis, which is a subspace, as Theorem 0.2 says.

**Theorem 0.3** (The orthogonal complement of a column space is a null space)**.** *For any $m \times n$ matrix* $\mathbf{A}$*,*

$$\mathcal{C}(\mathbf{A})^\perp = \mathcal{N}(\mathbf{A}^\top).$$

Proof

*Proof.* Let $\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_n$ be the columns of $\mathbf{A}$, so $\underset{\sim}{a}_j = (a_{1j}, \dots, a_{mj})$. For $\underset{\sim}{x} \in \mathbb{R}^m$, entry $j$ of $\mathbf{A}^\top\underset{\sim}{x}$ is

$$\begin{aligned}
(\mathbf{A}^\top\underset{\sim}{x})_j &= \sum_{i=1}^{m}(\mathbf{A}^\top)_{ji}\,x_i && \text{(Definition 8 in Matrix Multiplication)}\\
&= \sum_{i=1}^{m} a_{ij}\,x_i && \text{(Definition 6 in Matrices)}\\
&= \underset{\sim}{a}_j \cdot \underset{\sim}{x}. && \text{(Definition 10 in Vectors)}
\end{aligned}$$

So $\underset{\sim}{x} \in \mathcal{N}(\mathbf{A}^\top)$ (Definition 3 in Column Space and Null Space) exactly when $\underset{\sim}{a}_j \cdot \underset{\sim}{x} = 0$ for every column $\underset{\sim}{a}_j$.

$\mathcal{C}(\mathbf{A})^\perp \subseteq \mathcal{N}(\mathbf{A}^\top)$**.** Each column $\underset{\sim}{a}_j$ is in $\mathcal{C}(\mathbf{A})$ (Theorem 1 in Column Space and Null Space), so a vector orthogonal to all of $\mathcal{C}(\mathbf{A})$ is orthogonal to every column.

$\mathcal{N}(\mathbf{A}^\top) \subseteq \mathcal{C}(\mathbf{A})^\perp$**.** Suppose $\underset{\sim}{a}_j \cdot \underset{\sim}{x} = 0$ for every $j$, and take any $y \in \mathcal{C}(\mathbf{A})$, so $\underset{\sim}{y} = \mathbf{A}\underset{\sim}{z}$ for some $\underset{\sim}{z} \in \mathbb{R}^n$ (Definition 1 in Column Space and Null Space). Then

$$
\begin{aligned}
\underset{\sim}{y} \cdot \underset{\sim}{x} &= \sum_{i=1}^{m} (\mathbf{A}\underset{\sim}{z})_i \, x_i && \text{(Definition 10 in Vectors)} \\
&= \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} z_j \right) x_i && \text{(Definition 8 in Matrix Multiplication)} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \, z_j \, x_i && \text{(distribute each } x_i \text{ over the inner sum)} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \, z_j \, x_i && \text{(swap the order of the finite sums)} \\
&= \sum_{j=1}^{n} z_j \sum_{i=1}^{m} a_{ij} \, x_i && \text{(commute, then factor } z_j \text{ out of the inner sum)} \\
&= \sum_{j=1}^{n} z_j \, (\underset{\sim}{a}_j \cdot \underset{\sim}{x}) && \text{(Definition 10 in Vectors)} \\
&= \sum_{j=1}^{n} z_j \cdot 0 && \text{(each } \underset{\sim}{a}_j \cdot \underset{\sim}{x} = 0\text{)} \\
&= 0. && \text{(arithmetic)}
\end{aligned}
$$

So $\underset{\sim}{x}$ is orthogonal to every vector of $\mathcal{C}(\mathbf{A})$. □

**Example 0.4** (The complement of the column space in Example 1 in Column Space and Null Space)**.** For $\mathbf{A}$ in Example 1 in Column Space and Null Space, $\mathcal{C}(\mathbf{A}) = \text{span}\{(1, 3)\}$ (1). Example 3 in Column Space and Null Space found $\mathcal{N}(\mathbf{A}^\top) = \text{span}\{(-3, 1)\}$, so by Theorem 0.3 this line is $\mathcal{C}(\mathbf{A})^\perp$. Directly, for any $c$,

$$
\begin{aligned}
(c\,(1, 3)) \cdot (-3, 1) &= c\,((1, 3) \cdot (-3, 1)) && \text{(Theorem 0.1, special case } b = 0\text{, first slot)} \\
&= c\,(-3 + 3) && \text{(Definition 10 in Vectors)} \\
&= 0. && \text{(arithmetic)}
\end{aligned}
$$

**Exercise 0.4** (Finding the orthogonal complement of a column space)**.** Theorem 0.3 turns the orthogonal complement of a column space into a null space, which a linear system describes. Let

$$
\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.
$$

Find $\mathcal{C}(\mathbf{A})^\perp$ as a span.

**Hint**

Solve $\mathbf{A}^\top \underset{\sim}{y} = \underset{\sim}{0}_2$.

Solution

*Solution* 0.4. By Theorem 0.3, $\mathcal{C}(\mathbf{A})^\perp = \mathcal{N}(\mathbf{A}^\top)$. For $\underset{\sim}{y} = (y_1, y_2, y_3)$,

$$\mathbf{A}^\top \underset{\sim}{y} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 + y_3 \\ y_2 + y_3 \end{bmatrix},$$

which is $\underset{\sim}{0}_2$ exactly when $y_1 = y_2 = -y_3$. So

$$\mathcal{C}(\mathbf{A})^\perp = \text{span}\{(-1, -1, 1)\}.$$

As a check, $(-1, -1, 1)$ is orthogonal to both columns of $\mathbf{A}$: $-1+0+1 = 0$ and $0-1+1 = 0$.

**Theorem 0.4** (A subspace and its orthogonal complement make up the whole space)**.** *Let $\mathcal{S}$ be a subspace of $\mathbb{R}^p$. Then*

1. $\mathcal{S} \cap \mathcal{S}^\perp = \{\underset{\sim}{0}\}$;
2. $\dim(\mathcal{S}) + \dim(\mathcal{S}^\perp) = p$;
3. $\mathbb{R}^p = \mathcal{S} \oplus \mathcal{S}^\perp$ *(Definition 3 in Sums and Direct Sums).*

*So every $\underset{\sim}{y} \in \mathbb{R}^p$ can be written as $\underset{\sim}{y} = \underset{\sim}{u} + \underset{\sim}{v}$ with $\underset{\sim}{u} \in \mathcal{S}$ and $\underset{\sim}{v} \in \mathcal{S}^\perp$ in only one way (Theorem 4 in Sums and Direct Sums).*

Proof

*Proof.* $\mathcal{S}^\perp$ is a subspace (Theorem 0.2), so it has a dimension (Definition 4 in Bases and Dimension).

**Part 1.** Both $\mathcal{S}$ and $\mathcal{S}^\perp$ contain $\underset{\sim}{0}$ (Theorem 1 in Subspaces and Rank). If $\underset{\sim}{x}$ is in both, then $\underset{\sim}{x}$ is orthogonal to every vector of $\mathcal{S}$, $\underset{\sim}{x}$ itself included, so

$$\begin{aligned} \underset{\sim}{x} \cdot \underset{\sim}{x} &= x_1^2 + \cdots + x_p^2 \\ &= 0, \end{aligned}$$

which forces every $x_i = 0$.

**Part 2.** Let $d = \dim(\mathcal{S})$. If $d = 0$, then $\mathcal{S}$ has a basis (Theorem 4 in Bases and Dimension) with 0 vectors (Definition 4 in Bases and Dimension), the empty list, whose span is $\{\underset{\sim}{0}\}$ (Definition 5 in Subspaces and Rank); so $\mathcal{S} = \{\underset{\sim}{0}\}$, $\mathcal{S}^\perp = \mathbb{R}^p$ (every $\underset{\sim}{u}$ satisfies $\underset{\sim}{0} \cdot \underset{\sim}{u} = 0$), and $0 + p = p$ (Example 6 in Bases and Dimension). If $d \geq 1$, choose a basis $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_d$ of $\mathcal{S}$ (Theorem 4 in Bases and Dimension), and let $\mathbf{A}$ be the $p \times d$ matrix with

these columns. The basis spans $\mathcal{S}$, so $\mathcal{S} = \mathcal{C}(\mathbf{A})$ (Theorem 1 in Column Space and Null Space), and $\mathcal{S}^\perp = \mathcal{N}(\mathbf{A}^\top)$ (Theorem 0.3). $\mathbf{A}^\top$ is $d \times p$, so

$$\begin{aligned}
\dim(\mathcal{S}^\perp) &= \dim(\mathcal{N}(\mathbf{A}^\top)) && \text{(Theorem 0.3)} \\
&= \text{nullity}(\mathbf{A}^\top) && \text{(Definition 1 in Rank-Nullity and Rank Factorization)} \\
&= p - \text{rank}(\mathbf{A}^\top) && \text{(Theorem 2 in Rank-Nullity and Rank Factorization, for the } p \text{ columns of } \mathbf{A}^\top) \\
&= p - \text{rank}(\mathbf{A}) && \text{(Theorem 4 in Rank-Nullity and Rank Factorization)} \\
&= p - \dim(\mathcal{C}(\mathbf{A})) && \text{(Theorem 1 in Rank-Nullity and Rank Factorization)} \\
&= p - d. && (\mathcal{C}(\mathbf{A}) = \mathcal{S})
\end{aligned}$$

**Part 3.** $\mathcal{S} + \mathcal{S}^\perp$ is a subspace of $\mathbb{R}^p$ (Theorem 1 in Sums and Direct Sums), and

$$\begin{aligned}
\dim(\mathcal{S} + \mathcal{S}^\perp) &= \dim(\mathcal{S}) + \dim(\mathcal{S}^\perp) - \dim(\mathcal{S} \cap \mathcal{S}^\perp) && \text{(Theorem 2 in Sums and Direct Sums)} \\
&= p - \dim(\mathcal{S} \cap \mathcal{S}^\perp) && \text{(part 2)} \\
&= p - 0 && \text{(part 1, and } \dim(\{\underset{\sim}{0}\}) = 0 \text{ by Example 6 in Bases and Di} \\
&= p. && \text{(arithmetic)}
\end{aligned}$$

Since $\mathcal{S} + \mathcal{S}^\perp \subseteq \mathbb{R}^p$, $\mathbb{R}^p$ is a subspace (Example 3 in Subspaces and Rank) and $\dim(\mathbb{R}^p) = p$ (Example 6 in Bases and Dimension), Theorem 6 in Bases and Dimension gives $\mathcal{S} + \mathcal{S}^\perp = \mathbb{R}^p$. Together with part 1, $\mathbb{R}^p = \mathcal{S} \oplus \mathcal{S}^\perp$. The uniqueness of the split then follows from Theorem 4 in Sums and Direct Sums (statement 1 implies statement 3). □

**Example 0.5** (Splitting a vector of $\mathbb{R}^3$ along a line and its complement)**.** Let $\mathcal{S} = \text{span}\{(1,1,0)\}$ in $\mathbb{R}^3$. A single nonzero vector is linearly independent ($c\,\underset{\sim}{v} = \underset{\sim}{0}$ with $\underset{\sim}{v} \neq \underset{\sim}{0}$ forces $c = 0$), so $(1,1,0)$ is a basis of $\mathcal{S}$ and $\dim(\mathcal{S}) = 1$. A vector $\underset{\sim}{u}$ is in $\mathcal{S}^\perp$ exactly when $u_1 + u_2 = 0$: that condition says $(1,1,0) \cdot \underset{\sim}{u} = 0$, and then

$$\begin{aligned}
c\,(1,1,0) \cdot \underset{\sim}{u} &= c\,(u_1 + u_2) \\
&= 0
\end{aligned}$$

for every $c$. So

$$\begin{aligned}
\mathcal{S}^\perp &= \{(a, -a, c) : a, c \in \mathbb{R}\} \\
&= \text{span}\{(1,-1,0), (0,0,1)\}.
\end{aligned}$$

Those two vectors are linearly independent, because $a\,(1,-1,0) + c\,(0,0,1) = (a,-a,c)$ is $\underset{\sim}{0}$ only if

$$\begin{aligned} a &= c \\ &= 0, \end{aligned}$$

so $\dim(\mathcal{S}^\perp) = 2$, and $1 + 2 = 3$, as part 2 says.
To split $\underset{\sim}{y} = (3, 1, 2)$, look for $\underset{\sim}{u} = t\,(1, 1, 0)$ with

$$\begin{aligned} \underset{\sim}{v} &= \underset{\sim}{y} - \underset{\sim}{u} \\ &= (3 - t, 1 - t, 2) \end{aligned}$$

in $\mathcal{S}^\perp$: that membership needs $(3 - t) + (1 - t) = 0$, so $t = 2$. Then $\underset{\sim}{u} = (2, 2, 0)$ and $\underset{\sim}{v} = (1, -1, 2)$, with $\underset{\sim}{u} + \underset{\sim}{v} = (3, 1, 2)$ and

$$\begin{aligned} (1, 1, 0) \cdot (1, -1, 2) &= 1 - 1 + 0 \\ &= 0. \end{aligned}$$

**Exercise 0.5** (Splitting a vector into a piece in a line and an orthogonal piece)**.** By Theorem 0.4, every vector of $\mathbb{R}^p$ splits in only one way into a piece in a subspace $\mathcal{S}$ and a piece in its orthogonal complement. Let $\mathcal{S} = \text{span}\{(1, 2)\}$, a line in $\mathbb{R}^2$, and let $\underset{\sim}{y} = (5, 0)$. Find $\underset{\sim}{u} \in \mathcal{S}$ and $\underset{\sim}{v} \in \mathcal{S}^\perp$ with $\underset{\sim}{y} = \underset{\sim}{u} + \underset{\sim}{v}$.

**Hint**

Take $\underset{\sim}{u} = t\,(1, 2)$, and choose $t$ so that $\underset{\sim}{y} - \underset{\sim}{u}$ is orthogonal to $(1, 2)$.

Solution

*Solution* 0.5. Take $\underset{\sim}{u} = t\,(1, 2)$. Then $\underset{\sim}{v} = \underset{\sim}{y} - \underset{\sim}{u} = (5 - t, -2t)$ is in $\mathcal{S}^\perp$ exactly when

$$(1, 2) \cdot (5 - t, -2t) = 5 - t - 4t = 5 - 5t = 0,$$

so $t = 1$. Then $\underset{\sim}{u} = (1, 2)$ and $\underset{\sim}{v} = (4, -2)$, with $\underset{\sim}{u} + \underset{\sim}{v} = (5, 0)$ and $(1, 2) \cdot (4, -2) = 4 - 4 = 0$.

**Theorem 0.5** (The orthogonal complement of the orthogonal complement)**.** *For any subspace $\mathcal{S}$ of $\mathbb{R}^p$,*

$$(\mathcal{S}^\perp)^\perp = \mathcal{S}.$$

Proof

*Proof.* $\mathcal{S} \subseteq (\mathcal{S}^\perp)^\perp$**.** Take $\underset{\sim}{s} \in \mathcal{S}$ and any $\underset{\sim}{v} \in \mathcal{S}^\perp$. By Definition 0.1, $\underset{\sim}{s} \cdot \underset{\sim}{v} = 0$, so $\underset{\sim}{v} \cdot \underset{\sim}{s} = 0$ too (Theorem 1 in Vectors). So $\underset{\sim}{s}$ is orthogonal to every vector of $\mathcal{S}^\perp$, that is, $\underset{\sim}{s} \in (\mathcal{S}^\perp)^\perp$.

**Equal dimensions.** $\mathcal{S}^\perp$ is a subspace (Theorem 0.2), so part 2 of Theorem 0.4 applies to it as well as to $\mathcal{S}$:

$$\begin{aligned}\dim((\mathcal{S}^\perp)^\perp) &= p - \dim(\mathcal{S}^\perp) && \text{(Theorem 0.4, applied to } \mathcal{S}^\perp\text{)}\\ &= p - (p - \dim(\mathcal{S})) && \text{(Theorem 0.4, applied to } \mathcal{S}\text{)}\\ &= \dim(\mathcal{S}). && \text{(arithmetic)}\end{aligned}$$

$(\mathcal{S}^\perp)^\perp$ is a subspace (Theorem 0.2) that contains $\mathcal{S}$ and has the same dimension, so it equals $\mathcal{S}$ (Theorem 6 in Bases and Dimension). □

**Example 0.6** (Back to the line)**.** In Example 0.5, $\mathcal{S} = \text{span}\{(1,1,0)\}$ and $\mathcal{S}^\perp = \text{span}\{(1,-1,0),(0,0,1)\}$. Put those two spanning vectors in the columns of the $3 \times 2$ matrix $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\mathcal{S}^\perp = \mathcal{C}(\mathbf{B})$ (Theorem 1 in Column Space and Null Space) and $(\mathcal{S}^\perp)^\perp = \mathcal{N}(\mathbf{B}^\top)$ (Theorem 0.3). Since $\mathbf{B}^\top \underset{\sim}{w} = (w_1 - w_2,\ w_3)$ (Definition 8 in Matrix Multiplication), $\underset{\sim}{w} \in (\mathcal{S}^\perp)^\perp$ exactly when $w_1 - w_2 = 0$ and $w_3 = 0$. So

$$\begin{aligned}(\mathcal{S}^\perp)^\perp &= \{(a,a,0) : a \in \mathbb{R}\}\\ &= \mathcal{S}.\end{aligned}$$

**Example 0.7** (A set that is not a subspace does not come back)**.** The one-element set $\mathcal{X} = \{(1,1,0)\}$ is not a subspace. Its orthogonal complement is the same as that of $\mathcal{S}$ in Example 0.6, since both are defined by the single condition $u_1 + u_2 = 0$, so $(\mathcal{X}^\perp)^\perp = \text{span}\{(1,1,0)\}$, which contains $(2,2,0) \notin \mathcal{X}$. Theorem 0.5 needs $\mathcal{S}$ to be a subspace.

**Exercise 0.6** (Taking the orthogonal complement twice)**.** Theorem 0.5 says that taking the orthogonal complement of a subspace twice gives back the subspace. Let $\mathcal{S} = \text{span}\{(1,2)\}$ in $\mathbb{R}^2$, whose orthogonal complement is $\mathcal{S}^\perp = \text{span}\{(-2,1)\}$.
Find $(\mathcal{S}^\perp)^\perp$ directly from $\mathcal{S}^\perp = \text{span}\{(-2,1)\}$.

**Hint**

A vector $\underset{\sim}{w}$ is orthogonal to every multiple of $(-2,1)$ exactly when $-2w_1 + w_2 = 0$.

Solution

*Solution* 0.6. A vector $\underset{\sim}{w}$ is in $(\mathcal{S}^\perp)^\perp$ exactly when $c\,(-2,1)\cdot \underset{\sim}{w} = c\,(-2w_1 + w_2) = 0$ for every $c$, that is, when $w_2 = 2w_1$. So

$$(\mathcal{S}^\perp)^\perp = \{(w_1, 2w_1) : w_1 \in \mathbb{R}\} = \text{span}\{(1,2)\} = \mathcal{S},$$

as Theorem 0.5 says.

## The fundamental theorem of linear algebra

**Theorem 0.1** (Fundamental theorem of linear algebra)**.** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix with* $\text{rank}(\mathbf{A}) = r$*. Then*

1. $\mathcal{C}(\mathbf{A})^\perp = \mathcal{N}(\mathbf{A}^\top)$*, and* $\mathbb{R}^m = \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\top)$*;*
2. $\mathcal{C}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^\top)^\perp$*;*
3. $\mathcal{N}(\mathbf{A})^\perp = \mathcal{C}(\mathbf{A}^\top)$*, and* $\mathbb{R}^n = \mathcal{C}(\mathbf{A}^\top) \oplus \mathcal{N}(\mathbf{A})$*;*
4. $$\begin{aligned}\dim(\mathcal{C}(\mathbf{A})) &= \dim(\mathcal{C}(\mathbf{A}^\top)) \\ &= r,\end{aligned}$$

   $\dim(\mathcal{N}(\mathbf{A})) = n - r$*, and* $\dim(\mathcal{N}(\mathbf{A}^\top)) = m - r$*.*

Proof

*Proof.* **Part 1.** The first equation is Theorem 0.3. $\mathcal{C}(\mathbf{A})$ is a subspace of $\mathbb{R}^m$ (Theorem 1 in Column Space and Null Space), so

$$\begin{aligned}\mathbb{R}^m &= \mathcal{C}(\mathbf{A}) \oplus \mathcal{C}(\mathbf{A})^\perp && \text{(Theorem 0.4)} \\ &= \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\top). && \text{(first equation)}\end{aligned}$$

**Part 2.**

$$\begin{aligned}\mathcal{C}(\mathbf{A}) &= (\mathcal{C}(\mathbf{A})^\perp)^\perp && \text{(Theorem 0.5)} \\ &= \mathcal{N}(\mathbf{A}^\top)^\perp. && \text{(part 1)}\end{aligned}$$

**Part 3.** Apply parts 2 and 1 to the $n \times m$ matrix $\mathbf{A}^\top$, using $(\mathbf{A}^\top)^\top = \mathbf{A}$: part 2 gives $\mathcal{C}(\mathbf{A}^\top) = \mathcal{N}(\mathbf{A})^\perp$, and part 1 gives $\mathbb{R}^n = \mathcal{C}(\mathbf{A}^\top) \oplus \mathcal{N}(\mathbf{A})$.

**Part 4.** $\dim(\mathcal{C}(\mathbf{A})) = r$ by Theorem 1 in Rank-Nullity and Rank Factorization. $\dim(\mathcal{C}(\mathbf{A}^\top)) = \operatorname{rank}(\mathbf{A}^\top)$ by Theorem 1 in Rank-Nullity and Rank Factorization, and that is $r$ by Theorem 4 in Rank-Nullity and Rank Factorization. $\mathbf{A}$ has $n$ columns, so $\dim(\mathcal{N}(\mathbf{A})) = n - r$ (Definition 1 in Rank-Nullity and Rank Factorization, Theorem 2 in Rank-Nullity and Rank Factorization). $\mathbf{A}^\top$ has $m$ columns and rank $r$, so $\dim(\mathcal{N}(\mathbf{A}^\top)) = m - r$ in the same way. □

**Example 0.1** (The four subspaces of the matrix in Example 1 in Column Space and Null Space)**.** For $\mathbf{A}$ in Example 1 in Column Space and Null Space, $m = 2$, $n = 3$ and $r = 1$ (Example 1 in Rank-Nullity and Rank Factorization).

- **In $\mathbb{R}^3$:** the row space $\mathcal{C}(\mathbf{A}^\top) = \operatorname{span}\{(1, -2, -2)\}$

  (1) has dimension $1 = r$, and $\mathcal{N}(\mathbf{A}) = \operatorname{span}\{(2, 1, 0), (2, 0, 1)\}$ has dimension $2 = n - r$ (Example 2 in Rank-Nullity and Rank Factorization). The spanning vectors are orthogonal:

$$\begin{aligned}(1, -2, -2) \cdot (2, 1, 0) &= 2 - 2 + 0 \\ &= 0\end{aligned}$$

  and

$$\begin{aligned}(1, -2, -2) \cdot (2, 0, 1) &= 2 + 0 - 2 \\ &= 0.\end{aligned}$$

- **In $\mathbb{R}^2$:** $\mathcal{C}(\mathbf{A}) = \operatorname{span}\{(1, 3)\}$ has dimension $1 = r$, and $\mathcal{N}(\mathbf{A}^\top) = \operatorname{span}\{(-3, 1)\}$ has dimension $1 = m - r$ (Example 0.4).

By part 3, $y = (1, 0, 0)$ splits uniquely into a row-space part $t\,(1, -2, -2)$ and a null-space part. The null-space part $y - t\,(1, -2, -2)$ lies in $\mathcal{N}(\mathbf{A})$ (part 3), and $\mathcal{N}(\mathbf{A}) = \mathcal{C}(\mathbf{A}^\top)^\perp$ (Theorem 0.3 applied to $\mathbf{A}^\top$, with $(\mathbf{A}^\top)^\top = \mathbf{A}$), so it must be orthogonal to $(1, -2, -2)$:

$$\begin{aligned}
0 &= (1, -2, -2) \cdot ((1, 0, 0) - t\,(1, -2, -2)) && \text{(orthogonality)} \\
&= (1, -2, -2) \cdot (1 \cdot (1, 0, 0) + (-t)\,(1, -2, -2)) && \text{(write the difference as a linear combination)} \\
&= 1 \cdot ((1, -2, -2) \cdot (1, 0, 0)) + (-t)\,((1, -2, -2) \cdot (1, -2, -2)) && \text{(Theorem 0.1)} \\
&= 1 \cdot 1 + (-t) \cdot 9 && \text{(Definition 10 in Vectors)} \\
&= 1 - 9t, && \text{(arithmetic)}
\end{aligned}$$

so $t = \frac{1}{9}$. The row-space part is $\left(\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9}\right)$ and the null-space part is $\left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9}\right)$; as a check, row 1 of $\mathbf{A}$ gives $\frac{8}{9} - 2 \cdot \frac{2}{9} - 2 \cdot \frac{2}{9} = 0$, and row 2 is 3 times row 1, so $\mathbf{A}$ sends the null-space part to $\underset{\sim}{0}_2$.
These four statements are the fundamental theorem of linear algebra [2].

**Exercise 0.1** (Testing whether a linear system has a solution through the left null space)**.** Part 2 of Theorem 0.1 says that $\mathcal{C}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^\top)^\perp$, so a linear system $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ has a solution exactly when $\underset{\sim}{b}$ is orthogonal to every vector of the left null space. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Decide whether $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ has a solution, by testing $\underset{\sim}{b}$ against $\mathcal{N}(\mathbf{A}^\top)$.

**Hint**

Find one nonzero vector of $\mathcal{N}(\mathbf{A}^\top)$ whose third entry is 0.

Solution

*Solution* 0.1. Here $\mathbf{A}^\top \underset{\sim}{y} = (y_1 + y_2, y_1 + y_2)$, so $(1, -1, 0)$ is in $\mathcal{N}(\mathbf{A}^\top)$. But

$$\underset{\sim}{b} \cdot (1, -1, 0) = 1 - 2 + 0 = -1 \neq 0,$$

so $\underset{\sim}{b}$ is not in $\mathcal{N}(\mathbf{A}^\top)^\perp = \mathcal{C}(\mathbf{A})$, and $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ has no solution. Directly, the first two entries of $\mathbf{A}\underset{\sim}{x}$ are both $x_1 + x_2$, while those of $\underset{\sim}{b}$ are 1 and 2.


[1] Wikipedia contributors, "Orthogonal complement — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Orthogonal_complement

[2] Wikipedia contributors, "Fundamental theorem of linear algebra — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Fundamental_theorem_of_linear_algebra