Generalized Inverses and Linear Systems

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Generalized inverses and linear systems

Source

This section is adapted from Zhou [1], used under the MIT License (see the license text in Section in Subspaces and Rank). The source defers the existence of the Moore-Penrose inverse to the singular value decomposition; the proof here builds it from a rank factorization (Theorem 7 in Rank-Nullity and Rank Factorization) instead, and the section adds the projection $\mathbf{X}\mathbf{G}\mathbf{X}^\top$ for any generalized inverse $\mathbf{G}$ of $\mathbf{X}^\top\mathbf{X}$, which the source does not cover. Theorem 0.7 restores the source's formulas for the Moore-Penrose inverse of a matrix of full column rank or full row rank, and the fourth equivalent statement in Theorem 0.10 restores the source's rank test for consistency. These parts of the source are not part of this section:

- the QR forms of the full-rank formulas
- $\operatorname{rank}(\mathbf{A}^-) \geq \operatorname{rank}(\mathbf{A})$, $\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{A}\mathbf{A}^-)$, $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}\mathbf{A}^-) = \operatorname{rank}(\mathbf{A}^-\mathbf{A})$, and $\operatorname{rank}(\mathbf{A}^+) = \operatorname{rank}(\mathbf{A})$
- the remarks that $\mathbf{A}\mathbf{A}^-$ and $\mathbf{A}^-\mathbf{A}$ are projections, oblique in general and orthogonal for $\mathbf{A}^+$
- the reflexive generalized inverse (conditions 1 and 2 alone), and the generalized inverses of $\mathbf{A}^\top$ and $k\mathbf{A}$
- the operation counts for the triangular solves, LU and QR, and the comparison of the Cholesky factorization with LU

Theorem 0.1 (A scalar factor passes through a matrix product)**.** *For an $m \times k$ matrix $\mathbf{A}$, a $k \times n$ matrix $\mathbf{B}$ and a number c,*

$$\begin{aligned}\mathbf{A}\,(c\,\mathbf{B}) &= c\,(\mathbf{A}\mathbf{B}) \\ &= (c\,\mathbf{A})\,\mathbf{B}.\end{aligned}$$

In particular, $\mathbf{A}(\mathbf{B}-\mathbf{C}) = \mathbf{AB}-\mathbf{AC}$ *and* $(\mathbf{B}-\mathbf{C})\mathbf{D} = \mathbf{BD}-\mathbf{CD}$ *for matrices of compatible sizes, where* $\mathbf{B}-\mathbf{C} = \mathbf{B}+(-1)\mathbf{C}$ *(Theorem 6 in Matrices, Definition 9 in Matrices).*

Proof

Proof. $\mathbf{A}(c\mathbf{B}) = c(\mathbf{AB})$**.** For each entry (i,j),

$$\begin{aligned}
(\mathbf{A}(c\mathbf{B}))_{ij} &= \sum_{l=1}^{k} a_{il}(c\mathbf{B})_{lj} && \text{(Definition 1 in Matrix Multiplication)}\\
&= \sum_{l=1}^{k} a_{il}(c\,b_{lj}) && \text{(Definition 9 in Matrices)}\\
&= \sum_{l=1}^{k} c(a_{il}\,b_{lj}) && \text{(reorder the product of numbers)}\\
&= c\sum_{l=1}^{k} a_{il}\,b_{lj} && \text{(factor } c \text{ out of the sum)}\\
&= c(\mathbf{AB})_{ij} && \text{(Definition 1 in Matrix Multiplication)}\\
&= (c(\mathbf{AB}))_{ij}. && \text{(Definition 9 in Matrices)}
\end{aligned}$$

$(c\mathbf{A})\mathbf{B} = c(\mathbf{AB})$**.** For each entry (i,j),

$$\begin{aligned}
((c\mathbf{A})\mathbf{B})_{ij} &= \sum_{l=1}^{k}(c\,a_{il})\,b_{lj} && \text{(Definition 1 in Matrix Multiplication, Definition 9 in Matrices)}\\
&= c\sum_{l=1}^{k} a_{il}\,b_{lj} && \text{(factor } c \text{ out of the sum)}\\
&= (c(\mathbf{AB}))_{ij}. && \text{(Definition 1 in Matrix Multiplication, Definition 9 in Matrices)}
\end{aligned}$$

Differences.

$$\begin{aligned}
\mathbf{A}(\mathbf{B}-\mathbf{C}) &= \mathbf{A}(\mathbf{B}+(-1)\mathbf{C}) && \text{(the difference as a sum)}\\
&= \mathbf{AB}+\mathbf{A}((-1)\mathbf{C}) && \text{(Theorem 2 in Matrix Multiplication)}\\
&= \mathbf{AB}+(-1)(\mathbf{AC}) && \text{(first part)}\\
&= \mathbf{AB}-\mathbf{AC}, && \text{(the difference as a sum)}
\end{aligned}$$

and $(\mathbf{B}-\mathbf{C})\mathbf{D} = \mathbf{BD}-\mathbf{CD}$ the same way, using the second law in Theorem 2 in Matrix Multiplication and the second part. A column vector is a matrix with one column, so all of this applies with $\mathbf{B}$, $\mathbf{C}$ or $\mathbf{D}$ a vector. □

Example 0.1 (Pulling a scalar out of a product)**.** With $\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$:

$$\begin{aligned} \mathbf{A}\,(2\,\mathbf{B}) &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} \\ &= 22, \end{aligned}$$

and

$$\begin{aligned} 2\,(\mathbf{AB}) &= 2\,(3+8) \\ &= 22. \end{aligned}$$

Definition 0.1 (Generalized inverse)**.** A **generalized inverse** [2] of an $m \times n$ matrix $\mathbf{A}$ is an $n \times m$ matrix $\mathbf{G}$ such that

$$\underset{m\times n}{\underbrace{\mathbf{A}}}\ \underset{n\times m}{\underbrace{\mathbf{G}}}\ \underset{m\times n}{\underbrace{\mathbf{A}}} = \mathbf{A}.$$

Example 0.2 (A matrix with many generalized inverses)**.** Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{G} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{AGA} &= \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} && \text{(Definition 1 in Matrix Multiplication, for } \mathbf{AG}) \\ &= \begin{bmatrix} g_{11} & 0 \\ 0 & 0 \end{bmatrix}, && \text{(Definition 1 in Matrix Multiplication)} \end{aligned}$$

which equals $\mathbf{A}$ exactly when $g_{11} = 1$. So every matrix $\begin{bmatrix} 1 & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ is a generalized inverse of $\mathbf{A}$: a generalized inverse need not be unique. The zero matrix is not one, since its $(1,1)$ entry is 0.

Theorem 0.2 (An invertible matrix has only one generalized inverse)**.** *If* $\mathbf{A}$ *is an invertible* $p \times p$ *matrix (Definition 6 in Special Matrices, Quadratic Forms and Trace), its only generalized inverse is* $\mathbf{A}^{-1}$.

Proof

Proof.

$$\begin{aligned}\mathbf{A}\mathbf{A}^{-1}\mathbf{A} &= \mathbf{I}_p\mathbf{A}\\ &= \mathbf{A}\end{aligned}$$

(Definition 5 in Special Matrices, Quadratic Forms and Trace, Theorem 4 in Matrix Multiplication), so $\mathbf{A}^{-1}$ is a generalized inverse. If $\mathbf{G}$ is any generalized inverse, then

$$\begin{aligned}\mathbf{G} &= \mathbf{I}_p\,\mathbf{G}\,\mathbf{I}_p && \text{(Theorem 4 in Matrix Multiplication)}\\ &= (\mathbf{A}^{-1}\mathbf{A})\,\mathbf{G}\,(\mathbf{A}\mathbf{A}^{-1}) && \text{(Definition 5 in Special Matrices, Quadratic Forms and Trace)}\\ &= \mathbf{A}^{-1}\,(\mathbf{A}\mathbf{G}\mathbf{A})\,\mathbf{A}^{-1} && \text{(Theorem 1 in Matrix Multiplication)}\\ &= \mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1} && \text{(Definition 0.1)}\\ &= \mathbf{I}_p\,\mathbf{A}^{-1} && \text{(Definition 5 in Special Matrices, Quadratic Forms and Trace)}\\ &= \mathbf{A}^{-1}. && \text{(Theorem 4 in Matrix Multiplication)}\end{aligned}$$

□

Example 0.3 (The generalized inverse of an invertible matrix)**.** For $\mathbf{A} = \begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}$ of Example 5 in Special Matrices, Quadratic Forms and Trace, the only generalized inverse is $\mathbf{A}^{-1} = \begin{bmatrix}0.5 & -0.5\\ 0 & 1\end{bmatrix}$. The matrix of Example 0.2, by contrast, has infinitely many, so by Theorem 0.2 it is singular.

Theorem 0.3 (A generalized inverse solves every solvable system)**.** *Let* $\mathbf{A}$ *be* $m \times n$*, let* $\mathbf{G}$ *be a generalized inverse of* $\mathbf{A}$ *(Definition 0.1), and let* $\underset{\sim}{b} \in \mathbb{R}^m$*. If* $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ *has a solution, then* $\mathbf{G}\underset{\sim}{b}$ *is a solution.*

Proof

Proof. Let $\underset{\sim}{x}_0$ be a solution, so $\mathbf{A}\underset{\sim}{x}_0 = \underset{\sim}{b}$. Then

$$\begin{aligned}\mathbf{A}\,(\mathbf{G}\underset{\sim}{b}) &= \mathbf{A}\mathbf{G}\,(\mathbf{A}\underset{\sim}{x}_0) && (\text{substitute } \underset{\sim}{b} = \mathbf{A}\underset{\sim}{x}_0)\\ &= (\mathbf{A}\mathbf{G}\mathbf{A})\,\underset{\sim}{x}_0 && \text{(Theorem 1 in Matrix Multiplication)}\\ &= \mathbf{A}\underset{\sim}{x}_0 && \text{(Definition 0.1)}\\ &= \underset{\sim}{b}. && (\underset{\sim}{x}_0 \text{ is a solution})\end{aligned}$$

□

Example 0.4 (Solving with a generalized inverse)**.** With $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and its generalized inverse $\mathbf{G} = \begin{bmatrix} 1 & 5 \\ 7 & 0 \end{bmatrix}$ (Example 0.2), the system $\mathbf{A}\underset{\sim}{x} = (2, 0)$ has the solution $(2, 0)$, and $\mathbf{G}\,(2, 0) = (2, 14)$ is a solution too: $\mathbf{A}\,(2, 14) = (2, 0)$. The system $\mathbf{A}\underset{\sim}{x} = (2, 1)$ has no solution, since the second entry of $\mathbf{A}\underset{\sim}{x}$ is always 0; there $\mathbf{G}\,(2, 1) = (7, 14)$ gives $\mathbf{A}\,(7, 14) = (7, 0) \neq (2, 1)$.

Definition 0.2 (Moore-Penrose inverse)**.** A **Moore-Penrose inverse** [3] of an $m \times n$ matrix $\mathbf{A}$ is an $n \times m$ matrix $\mathbf{G}$ satisfying all four conditions

1. $\mathbf{AGA} = \mathbf{A}$;
2. $\mathbf{GAG} = \mathbf{G}$;
3. $(\mathbf{AG})^\top = \mathbf{AG}$;
4. $(\mathbf{GA})^\top = \mathbf{GA}$.

Example 0.5 (Checking the four conditions)**.** For $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the matrix $\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ satisfies all four:

$$\begin{aligned} \mathbf{AG} &= \mathbf{GA} \\ &= \mathbf{A} \end{aligned}$$

and $\mathbf{A}^2 = \mathbf{A}$, so conditions 1 and 2 read $\mathbf{A} = \mathbf{A}$, and conditions 3 and 4 hold because $\mathbf{A}$ is symmetric.

The generalized inverse $\mathbf{G}' = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ of $\mathbf{A}$ (Example 0.2) is not a Moore-Penrose inverse: $\mathbf{AG}' = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, which is not symmetric, so condition 3 fails.

Theorem 0.4 (A Moore-Penrose inverse is a generalized inverse)**.** *If* $\mathbf{G}$ *is a Moore-Penrose inverse of an* $m \times n$ *matrix* $\mathbf{A}$ *(Definition 0.2), then* $\mathbf{G}$ *is a generalized inverse of* $\mathbf{A}$ *(Definition 0.1).*

Proof

Proof. A generalized inverse of $\mathbf{A}$ is an $n \times m$ matrix $\mathbf{G}$ with $\mathbf{AGA} = \mathbf{A}$. A Moore-Penrose inverse is an $n \times m$ matrix, and condition 1 of Definition 0.2 is $\mathbf{AGA} = \mathbf{A}$. □

Example 0.6 (The Moore-Penrose inverse of a rank-one matrix)**.** The 3×2 matrix $\mathbf{A}$ below has rank 1, because its second column is twice its first, so it has no ordinary inverse. The function `MASS::ginv()` computes its Moore-Penrose inverse $\mathbf{G}$. The code checks the four conditions of Definition 0.2, of which the first is the equation that makes $\mathbf{G}$ a generalized inverse.

```
A <- cbind(c(1, 2, 3), c(2, 4, 6))
G <- MASS::ginv(A)
conditions <- c(
  "1: AGA = A" = isTRUE(all.equal(A %*% G %*% A, A)),
  "2: GAG = G" = isTRUE(all.equal(G %*% A %*% G, G)),
  "3: AG symmetric" = isTRUE(all.equal(A %*% G, t(A %*% G))),
  "4: GA symmetric" = isTRUE(all.equal(G %*% A, t(G %*% A)))
)
conditions
#>      1: AGA = A      2: GAG = G 3: AG symmetric 4: GA symmetric
#>             TRUE            TRUE            TRUE            TRUE
round(G, 4)
#>        [,1]   [,2]   [,3]
#> [1,] 0.0143 0.0286 0.0429
#> [2,] 0.0286 0.0571 0.0857
```

All 4 of 4 conditions hold, so $\mathbf{G}$ is a Moore-Penrose inverse of $\mathbf{A}$. By Theorem 0.4 it is also a generalized inverse, which is condition 1: $\mathbf{AGA} = \mathbf{A}$ up to rounding error.

Theorem 0.5 (A matrix has at most one Moore-Penrose inverse)**.** *If $\mathbf{G}_1$ and $\mathbf{G}_2$ are both Moore-Penrose inverses of an $m \times n$ matrix $\mathbf{A}$ (Definition 0.2), then $\mathbf{G}_1 = \mathbf{G}_2$.*

Proof

Proof. $\mathbf{AG}_1 = \mathbf{AG}_2$**.**

$$
\begin{aligned}
\mathbf{A}\mathbf{G}_1 &= (\mathbf{A}\mathbf{G}_1)^\top && \text{(condition 3 for } \mathbf{G}_1\text{)} \\
&= \mathbf{G}_1{}^\top \mathbf{A}^\top && \text{(Theorem 6 in Matrix Multiplication)} \\
&= \mathbf{G}_1{}^\top (\mathbf{A}\mathbf{G}_2\mathbf{A})^\top && \text{(condition 1 for } \mathbf{G}_2\text{)} \\
&= \mathbf{G}_1{}^\top \mathbf{A}^\top \mathbf{G}_2{}^\top \mathbf{A}^\top && \text{(Theorem 6 in Matrix Multiplication, twice)} \\
&= (\mathbf{G}_1{}^\top \mathbf{A}^\top)(\mathbf{G}_2{}^\top \mathbf{A}^\top) && \text{(Theorem 1 in Matrix Multiplication)} \\
&= (\mathbf{A}\mathbf{G}_1)^\top (\mathbf{A}\mathbf{G}_2)^\top && \text{(Theorem 6 in Matrix Multiplication, twice)} \\
&= (\mathbf{A}\mathbf{G}_1)(\mathbf{A}\mathbf{G}_2) && \text{(condition 3 for } \mathbf{G}_1 \text{ and } \mathbf{G}_2\text{)} \\
&= (\mathbf{A}\mathbf{G}_1\mathbf{A})\,\mathbf{G}_2 && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{A}\mathbf{G}_2. && \text{(condition 1 for } \mathbf{G}_1\text{)}
\end{aligned}
$$

$\mathbf{G}_1\mathbf{A} = \mathbf{G}_2\mathbf{A}$.

$$
\begin{aligned}
\mathbf{G}_1\mathbf{A} &= (\mathbf{G}_1\mathbf{A})^\top && \text{(condition 4 for } \mathbf{G}_1\text{)} \\
&= \mathbf{A}^\top \mathbf{G}_1{}^\top && \text{(Theorem 6 in Matrix Multiplication)} \\
&= (\mathbf{A}\mathbf{G}_2\mathbf{A})^\top \mathbf{G}_1{}^\top && \text{(condition 1 for } \mathbf{G}_2\text{)} \\
&= \mathbf{A}^\top \mathbf{G}_2{}^\top \mathbf{A}^\top \mathbf{G}_1{}^\top && \text{(Theorem 6 in Matrix Multiplication, twice)} \\
&= (\mathbf{A}^\top \mathbf{G}_2{}^\top)(\mathbf{A}^\top \mathbf{G}_1{}^\top) && \text{(Theorem 1 in Matrix Multiplication)} \\
&= (\mathbf{G}_2\mathbf{A})^\top (\mathbf{G}_1\mathbf{A})^\top && \text{(Theorem 6 in Matrix Multiplication, twice)} \\
&= (\mathbf{G}_2\mathbf{A})(\mathbf{G}_1\mathbf{A}) && \text{(condition 4 for } \mathbf{G}_2 \text{ and } \mathbf{G}_1\text{)} \\
&= \mathbf{G}_2\,(\mathbf{A}\mathbf{G}_1\mathbf{A}) && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{G}_2\mathbf{A}. && \text{(condition 1 for } \mathbf{G}_1\text{)}
\end{aligned}
$$

Conclusion.

$$
\begin{aligned}
\mathbf{G}_1 &= \mathbf{G}_1\mathbf{A}\mathbf{G}_1 && \text{(condition 2 for } \mathbf{G}_1\text{)} \\
&= \mathbf{G}_1\,(\mathbf{A}\mathbf{G}_1) && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{G}_1\,(\mathbf{A}\mathbf{G}_2) && \text{(the first step)} \\
&= (\mathbf{G}_1\mathbf{A})\,\mathbf{G}_2 && \text{(Theorem 1 in Matrix Multiplication)} \\
&= (\mathbf{G}_2\mathbf{A})\,\mathbf{G}_2 && \text{(the second step)} \\
&= \mathbf{G}_2\mathbf{A}\mathbf{G}_2 && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{G}_2. && \text{(condition 2 for } \mathbf{G}_2\text{)}
\end{aligned}
$$

□

Example 0.7 (Only one of the many generalized inverses qualifies)**.** The matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has infinitely many generalized inverses $\begin{bmatrix} 1 & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ (Example 0.2), and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

is a Moore-Penrose inverse (Example 0.5). By Theorem 0.5 it is the only one, so every other choice of g_{12}, g_{21}, g_{22} breaks condition 2, 3 or 4. For instance, $g_{21} = 1$ (the rest 0) gives $\mathbf{GA} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, which is not symmetric.

Theorem 0.6 (Every matrix has a Moore-Penrose inverse)**.** *Every $m \times n$ matrix $\mathbf{A}$ has a Moore-Penrose inverse (Definition 0.2), written $\mathbf{A}^+$; it is unique by Theorem 0.5.*

- *If $\mathbf{A} = \mathbf{0}_{m\times n}$, then $\mathbf{A}^+ = \mathbf{0}_{n\times m}$.*
- *If $\operatorname{rank}(\mathbf{A}) = r \geq 1$ and $\mathbf{A} = \mathbf{CR}$ is a rank factorization (Definition 2 in Rank-Nullity and Rank Factorization), then*

$$\underbrace{\mathbf{A}^+}_{n\times m} = \underbrace{\mathbf{R}^\top}_{n\times r} \underbrace{(\mathbf{R}\mathbf{R}^\top)^{-1}}_{r\times r} \underbrace{(\mathbf{C}^\top\mathbf{C})^{-1}}_{r\times r} \underbrace{\mathbf{C}^\top}_{r\times m}.$$

Proof

Proof. **The two cases cover every matrix.** If $\mathbf{A} \neq \mathbf{0}_{m\times n}$, it has a nonzero column, which is linearly independent on its own ($c\,\underset{\sim}{v} = \underset{\sim}{0}$ with $\underset{\sim}{v} \neq \underset{\sim}{0}$ forces $c = 0$), so $\operatorname{rank}(\mathbf{A}) \geq 1$ (Definition 2 in Subspaces and Rank).
The zero matrix. With $\mathbf{A} = \mathbf{0}_{m\times n}$ and $\mathbf{G} = \mathbf{0}_{n\times m}$, every product in conditions 1 to 4 is a zero matrix, and a zero matrix equals its own transpose, so all four conditions hold.
The two inverses exist. Now suppose $\operatorname{rank}(\mathbf{A}) = r \geq 1$. A rank factorization exists (Theorem 7 in Rank-Nullity and Rank Factorization). $\mathbf{C}$ is $m \times r$ and $\mathbf{R}$ is $r \times n$. Both have rank r:

$$\begin{aligned} r &= \operatorname{rank}(\mathbf{CR}) && (\mathbf{A} = \mathbf{CR}) \\ &\leq \operatorname{rank}(\mathbf{C}) && \text{(Theorem 3 in Rank-Nullity and Rank Factorization)} \\ &\leq r, && \text{(Corollary 1 in Rank-Nullity and Rank Factorization, } \mathbf{C} \text{ has } r \text{ columns)} \end{aligned}$$

and, using $(\mathbf{CR})^\top = \mathbf{R}^\top\mathbf{C}^\top$ (Theorem 6 in Matrix Multiplication),

$$\begin{aligned} r &= \operatorname{rank}(\mathbf{R}^\top\mathbf{C}^\top) && \text{(Theorem 4 in Rank-Nullity and Rank Factorization)} \\ &\leq \operatorname{rank}(\mathbf{R}^\top) && \text{(Theorem 3 in Rank-Nullity and Rank Factorization)} \\ &\leq r. && \text{(Corollary 1 in Rank-Nullity and Rank Factorization, } \mathbf{R}^\top \text{ has } r \text{ columns)} \end{aligned}$$

So $\mathbf{C}$ and $\mathbf{R}^\top$ have full column rank r, and $\mathbf{C}^\top\mathbf{C}$ and $\mathbf{RR}^\top = (\mathbf{R}^\top)^\top\mathbf{R}^\top$ are invertible (Theorem 1 in Design Matrices and Projections). They are symmetric:

$$
\begin{aligned}
(\mathbf{C}^\top\mathbf{C})^\top &= \mathbf{C}^\top(\mathbf{C}^\top)^\top \\
&= \mathbf{C}^\top\mathbf{C}
\end{aligned}
$$

(Theorem 6 in Matrix Multiplication, Definition 6 in Matrices), and likewise for $\mathbf{R}\mathbf{R}^\top$. Write $\mathbf{S} \stackrel{\text{def}}{=} (\mathbf{R}\mathbf{R}^\top)^{-1}$ and $\mathbf{T} \stackrel{\text{def}}{=} (\mathbf{C}^\top\mathbf{C})^{-1}$, both symmetric (Corollary 1 in Special Matrices, Quadratic Forms and Trace), and $\mathbf{G} \stackrel{\text{def}}{=} \mathbf{R}^\top\mathbf{S}\mathbf{T}\mathbf{C}^\top$.

Two products.

$$
\begin{aligned}
\mathbf{A}\mathbf{G} &= \mathbf{C}\mathbf{R}\,\mathbf{R}^\top\mathbf{S}\mathbf{T}\mathbf{C}^\top && \text{(substitute } \mathbf{A} \text{ and } \mathbf{G}\text{)} \\
&= \mathbf{C}\,(\mathbf{R}\mathbf{R}^\top)\,\mathbf{S}\,\mathbf{T}\mathbf{C}^\top && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{C}\,\mathbf{I}_r\,\mathbf{T}\mathbf{C}^\top && \text{(Definition 5 in Special Matrices, Quadratic Forms and Trace)} \\
&= \mathbf{C}\mathbf{T}\mathbf{C}^\top, && \text{(Theorem 4 in Matrix Multiplication)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{G}\mathbf{A} &= \mathbf{R}^\top\mathbf{S}\mathbf{T}\mathbf{C}^\top\,\mathbf{C}\mathbf{R} && \text{(substitute } \mathbf{G} \text{ and } \mathbf{A}\text{)} \\
&= \mathbf{R}^\top\mathbf{S}\,\mathbf{T}\,(\mathbf{C}^\top\mathbf{C})\,\mathbf{R} && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{R}^\top\mathbf{S}\,\mathbf{I}_r\,\mathbf{R} && \text{(Definition 5 in Special Matrices, Quadratic Forms and Trace)} \\
&= \mathbf{R}^\top\mathbf{S}\mathbf{R}. && \text{(Theorem 4 in Matrix Multiplication)}
\end{aligned}
$$

Conditions 3 and 4.

$$
\begin{aligned}
(\mathbf{A}\mathbf{G})^\top &= (\mathbf{C}\mathbf{T}\mathbf{C}^\top)^\top && \text{(first product)} \\
&= (\mathbf{C}\,(\mathbf{T}\mathbf{C}^\top))^\top && \text{(Theorem 1 in Matrix Multiplication)} \\
&= (\mathbf{T}\mathbf{C}^\top)^\top\,\mathbf{C}^\top && \text{(Theorem 6 in Matrix Multiplication)} \\
&= (\mathbf{C}^\top)^\top\,\mathbf{T}^\top\,\mathbf{C}^\top && \text{(Theorem 6 in Matrix Multiplication)} \\
&= \mathbf{C}\,\mathbf{T}^\top\,\mathbf{C}^\top && \text{(Definition 6 in Matrices)} \\
&= \mathbf{C}\mathbf{T}\mathbf{C}^\top, && (\mathbf{T} \text{ is symmetric})
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{G}\mathbf{A})^\top &= (\mathbf{R}^\top\mathbf{S}\mathbf{R})^\top && \text{(second product)} \\
&= (\mathbf{R}^\top\,(\mathbf{S}\mathbf{R}))^\top && \text{(Theorem 1 in Matrix Multiplication)} \\
&= (\mathbf{S}\mathbf{R})^\top\,(\mathbf{R}^\top)^\top && \text{(Theorem 6 in Matrix Multiplication)} \\
&= \mathbf{R}^\top\,\mathbf{S}^\top\,(\mathbf{R}^\top)^\top && \text{(Theorem 6 in Matrix Multiplication)} \\
&= \mathbf{R}^\top\,\mathbf{S}^\top\,\mathbf{R} && \text{(Definition 6 in Matrices)} \\
&= \mathbf{R}^\top\mathbf{S}\mathbf{R}. && (\mathbf{S} \text{ is symmetric})
\end{aligned}
$$

Condition 1.

$$
\begin{aligned}
\mathbf{AGA} &= \mathbf{CTC}^\top \mathbf{CR} && \text{(first product)} \\
&= \mathbf{C}\,\mathbf{T}\,(\mathbf{C}^\top\mathbf{C})\,\mathbf{R} && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{C}\,\mathbf{I}_r\,\mathbf{R} && \text{(Definition 5 in Special Matrices, Quadratic Forms and Trace)} \\
&= \mathbf{A}. && \text{(Theorem 4 in Matrix Multiplication)}
\end{aligned}
$$

Condition 2.

$$
\begin{aligned}
\mathbf{GAG} &= \mathbf{R}^\top\mathbf{SR}\,\mathbf{R}^\top\mathbf{STC}^\top && \text{(second product)} \\
&= \mathbf{R}^\top\,\mathbf{S}\,(\mathbf{RR}^\top)\,\mathbf{S}\,\mathbf{TC}^\top && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{R}^\top\,\mathbf{I}_r\,\mathbf{S}\,\mathbf{TC}^\top && \text{(Definition 5 in Special Matrices, Quadratic Forms and Trace)} \\
&= \mathbf{G}. && \text{(Theorem 4 in Matrix Multiplication)}
\end{aligned}
$$

□

Example 0.8 (The Moore-Penrose inverse of a rank-one matrix)**.** For $\mathbf{A}$ in Example 1 in Column Space and Null Space, the rank factorization $\mathbf{C} = \begin{bmatrix}1\\3\end{bmatrix}$, $\mathbf{R} = \begin{bmatrix}1 & -2 & -2\end{bmatrix}$ (Example 11 in Rank-Nullity and Rank Factorization) gives

$$
\begin{aligned}
\mathbf{C}^\top\mathbf{C} &= 1 + 9 \\
&= 10
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{RR}^\top &= 1 + 4 + 4 \\
&= 9,
\end{aligned}
$$

so

$$
\begin{aligned}
\mathbf{A}^+ &= \frac{1}{9\cdot 10}\begin{bmatrix}1\\-2\\-2\end{bmatrix}\begin{bmatrix}1 & 3\end{bmatrix} \\
&= \frac{1}{90}\begin{bmatrix}1 & 3\\-2 & -6\\-2 & -6\end{bmatrix}.
\end{aligned}
$$

As a check on condition 1,

$$
\begin{aligned}
\mathbf{AA}^+ &= \mathbf{CTC}^\top \\
&= \frac{1}{10}\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix},
\end{aligned}
$$

and its first row times $\mathbf{A}$ is $\frac{1}{10}(1 \cdot (1,-2,-2) + 3 \cdot (3,-6,-6)) = (1,-2,-2)$, the first row of $\mathbf{A}$; the second row is 3 times the first, like the second row of $\mathbf{A}$.

Theorem 0.7 (The Moore-Penrose inverse of a matrix of full column rank or full row rank)**.** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix.*

1. *If* $\mathbf{A}$ *has full column rank (Definition 3 in Subspaces and Rank), then* $\mathbf{A}^\top\mathbf{A}$ *is invertible and* $\mathbf{A}^+ = \left((\mathbf{A}^\top\mathbf{A})\right)^{-1}\mathbf{A}^\top$.
2. *If* $\text{rank}(\mathbf{A}) = m$*, then* $\mathbf{A}\mathbf{A}^\top$ *is invertible and* $\mathbf{A}^+ = \mathbf{A}^\top\left((\mathbf{A}\mathbf{A}^\top)\right)^{-1}$.

Proof

Proof. By Theorem 0.5, it is enough to show that the matrix $\mathbf{G}$ named in each part satisfies the four conditions of Definition 0.2.

Part 1. $\mathbf{A}^\top\mathbf{A}$ is invertible by Theorem 1 in Design Matrices and Projections. Let $\mathbf{G} = \left((\mathbf{A}^\top\mathbf{A})\right)^{-1}\mathbf{A}^\top$. Then

$$\begin{aligned}\mathbf{G}\mathbf{A} &= \left((\mathbf{A}^\top\mathbf{A})\right)^{-1}\mathbf{A}^\top\mathbf{A} && \text{(definition of } \mathbf{G}\text{)}\\ &= \mathbf{I}_n. && \text{(Definition 5 in Special Matrices, Quadratic Forms and Trace)}\end{aligned}$$

- Condition 4: $(\mathbf{G}\mathbf{A})^\top = {\mathbf{I}_n}^\top = \mathbf{I}_n = \mathbf{G}\mathbf{A}$.
- Condition 1: $\mathbf{A}\mathbf{G}\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}$.
- Condition 2: $\mathbf{G}\mathbf{A}\mathbf{G} = \mathbf{I}_n\mathbf{G} = \mathbf{G}$.
- Condition 3: $(\mathbf{A}\mathbf{G})^\top = \mathbf{A}\left(\left((\mathbf{A}^\top\mathbf{A})\right)^{-1}\right)^\top\mathbf{A}^\top = \mathbf{A}\left((\mathbf{A}^\top\mathbf{A})^\top\right)^{-1}\mathbf{A}^\top = \mathbf{A}\left((\mathbf{A}^\top\mathbf{A})\right)^{-1}\mathbf{A}^\top = \mathbf{A}\mathbf{G}$, by Theorem 6 in Matrix Multiplication, Theorem 2 in Special Matrices, Quadratic Forms and Trace, and because $\mathbf{A}^\top\mathbf{A}$ is symmetric.

Part 2. $\text{rank}(\mathbf{A}\mathbf{A}^\top) = \text{rank}(\mathbf{A}) = m$ (Theorem 4 in Rank-Nullity and Rank Factorization), and $\mathbf{A}\mathbf{A}^\top$ is $m \times m$, so it is invertible (Theorem 1 in Similarity and Diagonalization). Let $\mathbf{G} = \mathbf{A}^\top\left((\mathbf{A}\mathbf{A}^\top)\right)^{-1}$. Then $\mathbf{A}\mathbf{G} = \mathbf{A}\mathbf{A}^\top\left((\mathbf{A}\mathbf{A}^\top)\right)^{-1} = \mathbf{I}_m$ (Definition 5 in Special Matrices, Quadratic Forms and Trace).

- Condition 3: $(\mathbf{A}\mathbf{G})^\top = {\mathbf{I}_m}^\top = \mathbf{I}_m = \mathbf{A}\mathbf{G}$.
- Condition 1: $\mathbf{A}\mathbf{G}\mathbf{A} = \mathbf{I}_m\mathbf{A} = \mathbf{A}$.
- Condition 2: $\mathbf{G}\mathbf{A}\mathbf{G} = \mathbf{G}\mathbf{I}_m = \mathbf{G}$.
- Condition 4: $(\mathbf{G}\mathbf{A})^\top = \mathbf{A}^\top\left(\left((\mathbf{A}\mathbf{A}^\top)\right)^{-1}\right)^\top\mathbf{A} = \mathbf{A}^\top\left((\mathbf{A}\mathbf{A}^\top)\right)^{-1}\mathbf{A} = \mathbf{G}\mathbf{A}$, by Theorem 6 in Matrix Multiplication, Theorem 2 in Special Matrices, Quadratic Forms and Trace, and because $\mathbf{A}\mathbf{A}^\top$ is symmetric.

□

Example 0.9 (Moore-Penrose inverses of a column and a row)**.** The 2×1 matrix $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has full column rank, and $\mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 2 \end{bmatrix}$, so

$$\begin{aligned} \mathbf{A}^+ &= \tfrac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Then $\mathbf{A}^+\mathbf{A} = \begin{bmatrix} 1 \end{bmatrix}$, and $\mathbf{A}\mathbf{A}^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is symmetric.

The 1×2 matrix $\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ has rank 1, which is its count of rows, and $\mathbf{B}\mathbf{B}^\top = \begin{bmatrix} 2 \end{bmatrix}$, so

$$\begin{aligned} \mathbf{B}^+ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tfrac{1}{2} \\ &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Theorem 0.8 (The Moore-Penrose inverse of a symmetric matrix from its eigendecomposition)**.** *Let $\mathbf{S} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\top$ be an eigendecomposition (Definition 3 in Eigenvalues and Matrix Decompositions) of a symmetric matrix $\mathbf{S}$, with diagonal entries $\lambda_1, \ldots, \lambda_p$ in the diagonal matrix $\mathbf{\Lambda}$. Let $\mathbf{\Lambda}^+$ be the diagonal matrix with jth diagonal entry $1/\lambda_j$ if $\lambda_j \neq 0$ and 0 if $\lambda_j = 0$. Then*

$$\mathbf{S}^+ = \mathbf{Q}\mathbf{\Lambda}^+\mathbf{Q}^\top.$$

Proof

Proof. Let $\mathbf{G} = \mathbf{Q}\mathbf{\Lambda}^+\mathbf{Q}^\top$. By Theorem 0.5, it is enough to show that $\mathbf{G}$ satisfies the four conditions of Definition 0.2 for $\mathbf{S}$.

Two facts about the diagonal matrices. The jth diagonal entry of $\mathbf{\Lambda}\mathbf{\Lambda}^+\mathbf{\Lambda}$ is $\lambda_j \cdot (1/\lambda_j) \cdot \lambda_j = \lambda_j$ if $\lambda_j \neq 0$, and $0 = \lambda_j$ if $\lambda_j = 0$. So $\mathbf{\Lambda}\mathbf{\Lambda}^+\mathbf{\Lambda} = \mathbf{\Lambda}$. In the same way, $\mathbf{\Lambda}^+\mathbf{\Lambda}\mathbf{\Lambda}^+ = \mathbf{\Lambda}^+$. Both $\mathbf{\Lambda}\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^+\mathbf{\Lambda}$ are diagonal, so each is symmetric.

Conditions 1 and 2.

$$\begin{aligned} \mathbf{S}\mathbf{G}\mathbf{S} &= \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\top \mathbf{Q}\mathbf{\Lambda}^+\mathbf{Q}^\top \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\top && \text{(definitions of } \mathbf{S} \text{ and } \mathbf{G}) \\ &= \mathbf{Q}\mathbf{\Lambda}\mathbf{\Lambda}^+\mathbf{\Lambda}\mathbf{Q}^\top && (\mathbf{Q}^\top\mathbf{Q} = \mathbf{I}, \text{ twice}) \\ &= \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\top && (\mathbf{\Lambda}\mathbf{\Lambda}^+\mathbf{\Lambda} = \mathbf{\Lambda}) \\ &= \mathbf{S}, && \text{(definition of } \mathbf{S}) \end{aligned}$$

and likewise

$$\begin{aligned}\mathbf{GSG} &= \mathbf{Q}\mathbf{\Lambda}^{+}\mathbf{\Lambda}\mathbf{\Lambda}^{+}\mathbf{Q}^{\top} && (\mathbf{Q}^{\top}\mathbf{Q}=\mathbf{I},\ \text{twice})\\ &= \mathbf{Q}\mathbf{\Lambda}^{+}\mathbf{Q}^{\top} && (\mathbf{\Lambda}^{+}\mathbf{\Lambda}\mathbf{\Lambda}^{+}=\mathbf{\Lambda}^{+})\\ &= \mathbf{G}. && (\text{definition of } \mathbf{G})\end{aligned}$$

Conditions 3 and 4.

$$\begin{aligned}\mathbf{SG} &= \mathbf{Q}\mathbf{\Lambda}\mathbf{\Lambda}^{+}\mathbf{Q}^{\top} && (\mathbf{Q}^{\top}\mathbf{Q}=\mathbf{I})\\ &= (\mathbf{Q}\mathbf{\Lambda}\mathbf{\Lambda}^{+}\mathbf{Q}^{\top})^{\top} && (\mathbf{\Lambda}\mathbf{\Lambda}^{+} \text{ is diagonal, so equals its transpose})\\ &= (\mathbf{SG})^{\top}, && (\text{just shown, in the other order})\end{aligned}$$

and the same steps with $\mathbf{\Lambda}^{+}\mathbf{\Lambda}$ in place of $\mathbf{\Lambda}\mathbf{\Lambda}^{+}$ give $\mathbf{GS} = (\mathbf{GS})^{\top}$. □

Example 0.10 (The Moore-Penrose inverse of a rank-one symmetric matrix)**.** For the symmetric matrix $\mathbf{S} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, an eigendecomposition is

$$\mathbf{S} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}}_{\mathbf{Q}}\underbrace{\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}}_{\mathbf{\Lambda}}\mathbf{Q}^{\top}.$$

Inverting only the nonzero eigenvalue gives $\mathbf{\Lambda}^{+} = \begin{bmatrix}1/2 & 0\\ 0 & 0\end{bmatrix}$, so

$$\begin{aligned}\mathbf{S}^{+} &= \mathbf{Q}\mathbf{\Lambda}^{+}\mathbf{Q}^{\top} && (\text{Theorem 0.8})\\ &= \frac{1}{2}\cdot\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} && (\text{only the first column of } \mathbf{Q} \text{ contributes})\\ &= \frac{1}{4}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.\end{aligned}$$

Condition 1 of Definition 0.2 holds: $\mathbf{SS}^{+} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, and multiplying by $\mathbf{S}$ again gives $\mathbf{S}$. An inverse of $\mathbf{S}$ itself does not exist, because $\mathbf{S}$ has a zero eigenvalue.

Theorem 0.9 (The Moore-Penrose inverse of a Gram matrix)**.** *For every $m \times n$ matrix $\mathbf{A}$,*

$$(\mathbf{A}^{\top}\mathbf{A})^{+} = \mathbf{A}^{+}(\mathbf{A}^{+})^{\top}.$$

Proof

Proof. Let $\mathbf{M} = \mathbf{A}^\top \mathbf{A}$ and $\mathbf{G} = \mathbf{A}^+ (\mathbf{A}^+)^\top$. By Theorem 0.5, it is enough to show that $\mathbf{G}$ satisfies the four conditions of Definition 0.2 for $\mathbf{M}$.

Two identities. Conditions 3 and 4 for $\mathbf{A}^+$ say that $\mathbf{A}\mathbf{A}^+$ and $\mathbf{A}^+\mathbf{A}$ are symmetric. Using condition 1 for $\mathbf{A}^+$,

$$\begin{aligned} \mathbf{A}^\top \mathbf{A}\mathbf{A}^+ &= \mathbf{A}^\top (\mathbf{A}\mathbf{A}^+)^\top && \text{(condition 3 for } \mathbf{A}^+) \\ &= (\mathbf{A}\mathbf{A}^+\mathbf{A})^\top && \text{(Theorem 6 in Matrix Multiplication)} \\ &= \mathbf{A}^\top. && \text{(condition 1 for } \mathbf{A}^+) \end{aligned}$$

So

$$\begin{aligned} \mathbf{M}\mathbf{G} &= \mathbf{A}^\top \mathbf{A}\mathbf{A}^+ (\mathbf{A}^+)^\top && \text{(definitions of } \mathbf{M} \text{ and } \mathbf{G}) \\ &= \mathbf{A}^\top (\mathbf{A}^+)^\top && (\mathbf{A}^\top \mathbf{A}\mathbf{A}^+ = \mathbf{A}^\top) \\ &= (\mathbf{A}^+\mathbf{A})^\top, && \text{(Theorem 6 in Matrix Multiplication)} \end{aligned}$$

which is $\mathbf{A}^+\mathbf{A}$ by condition 4 for $\mathbf{A}^+$. Hence $\mathbf{M}\mathbf{G} = \mathbf{A}^+\mathbf{A}$, and it is symmetric, which is condition 3 for $\mathbf{G}$.

For the other order,

$$\begin{aligned} \mathbf{G}\mathbf{M} &= \mathbf{A}^+ (\mathbf{A}^+)^\top \mathbf{A}^\top \mathbf{A} && \text{(definitions of } \mathbf{G} \text{ and } \mathbf{M}) \\ &= \mathbf{A}^+ (\mathbf{A}\mathbf{A}^+)^\top \mathbf{A} && \text{(Theorem 6 in Matrix Multiplication)} \\ &= \mathbf{A}^+\mathbf{A}\mathbf{A}^+\mathbf{A} && \text{(condition 3 for } \mathbf{A}^+) \\ &= \mathbf{A}^+\mathbf{A}, && \text{(condition 1 for } \mathbf{A}^+) \end{aligned}$$

so $\mathbf{G}\mathbf{M} = \mathbf{A}^+\mathbf{A}$ is symmetric, which is condition 4 for $\mathbf{G}$.

Conditions 1 and 2.

$$\begin{aligned} \mathbf{M}\mathbf{G}\mathbf{M} &= \mathbf{A}^+\mathbf{A}\,\mathbf{A}^\top\mathbf{A} && (\mathbf{M}\mathbf{G} = \mathbf{A}^+\mathbf{A}) \\ &= (\mathbf{A}^+\mathbf{A})^\top \mathbf{A}^\top\mathbf{A} && \text{(condition 4 for } \mathbf{A}^+) \\ &= (\mathbf{A}\mathbf{A}^+\mathbf{A})^\top \mathbf{A} && \text{(Theorem 6 in Matrix Multiplication)} \\ &= \mathbf{A}^\top\mathbf{A} && \text{(condition 1 for } \mathbf{A}^+) \\ &= \mathbf{M}, && \text{(definition of } \mathbf{M}) \\ \mathbf{G}\mathbf{M}\mathbf{G} &= \mathbf{A}^+\mathbf{A}\,\mathbf{A}^+ (\mathbf{A}^+)^\top && (\mathbf{G}\mathbf{M} = \mathbf{A}^+\mathbf{A}) \\ &= \mathbf{A}^+ (\mathbf{A}^+)^\top && \text{(condition 2 for } \mathbf{A}^+) \\ &= \mathbf{G}. && \text{(definition of } \mathbf{G}) \end{aligned}$$

□

Example 0.11 (The Moore-Penrose inverse of the Gram matrix of a column of ones)**.** For $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the Gram matrix is $\mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 2 \end{bmatrix}$, so $(\mathbf{A}^\top \mathbf{A})^+ = \begin{bmatrix} 1/2 \end{bmatrix}$. By Theorem 0.7, $\mathbf{A}^+ = ((\mathbf{A}^\top \mathbf{A}))^{-1} \mathbf{A}^\top = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{A}^+ (\mathbf{A}^+)^\top &= \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}, \end{aligned}$$

which is $(\mathbf{A}^\top \mathbf{A})^+$, as Theorem 0.9 states.

Corollary 0.1 (The Moore-Penrose inverse from the Gram matrix)**.** *For every $m \times n$ matrix* $\mathbf{A}$*,*

$$\mathbf{A}^+ = (\mathbf{A}^\top \mathbf{A})^+ \mathbf{A}^\top.$$

Proof

Proof.

$$\begin{aligned} (\mathbf{A}^\top \mathbf{A})^+ \mathbf{A}^\top &= \mathbf{A}^+ (\mathbf{A}^+)^\top \mathbf{A}^\top && \text{(Theorem 0.9)} \\ &= \mathbf{A}^+ (\mathbf{A}\mathbf{A}^+)^\top && \text{(Theorem 6 in Matrix Multiplication)} \\ &= \mathbf{A}^+ \mathbf{A}\mathbf{A}^+ && \text{(condition 3 of Definition 0.2)} \\ &= \mathbf{A}^+. && \text{(condition 2 of Definition 0.2)} \end{aligned}$$

□

Example 0.12 (Recovering the Moore-Penrose inverse of a column of ones)**.** For $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, Example 0.11 found $(\mathbf{A}^\top \mathbf{A})^+ = \begin{bmatrix} 1/2 \end{bmatrix}$. Then

$$\begin{aligned} (\mathbf{A}^\top \mathbf{A})^+ \mathbf{A}^\top &= \begin{bmatrix} 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}, \end{aligned}$$

which is $\mathbf{A}^+$, as Corollary 0.1 states.

Definition 0.3 (Consistent linear system)**.** A linear system (Definition 9 in Matrix Multiplication) $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$, with $\mathbf{A}$ an $m \times n$ matrix and $\underset{\sim}{b} \in \mathbb{R}^m$, is **consistent** [4] if it has at least one solution $\underset{\sim}{x} \in \mathbb{R}^n$, and **inconsistent** otherwise.

Example 0.13 (A consistent system and an inconsistent one)**.** With $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{A}(x_1, x_2) = (x_1, 0)$ (Definition 8 in Matrix Multiplication).

- $\mathbf{A}\underset{\sim}{x} = (2, 0)$ is consistent: $\underset{\sim}{x} = (2, 0)$ is a solution.
- $\mathbf{A}\underset{\sim}{x} = (2, 1)$ is inconsistent: the second entry of $\mathbf{A}\underset{\sim}{x}$ is always 0, never 1.

Theorem 0.10 (When a linear system has a solution)**.** *Let $\mathbf{A}$ be $m \times n$, let $\underset{\sim}{b} \in \mathbb{R}^m$, and let $\mathbf{G}$ be any generalized inverse of $\mathbf{A}$ (Definition 0.1). The following statements are equivalent:*

1. *$\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ is consistent (Definition 0.3);*
2. *$\underset{\sim}{b} \in \mathcal{C}(\mathbf{A})$ (Definition 1 in Column Space and Null Space);*
3. *$\mathbf{A}\mathbf{G}\underset{\sim}{b} = \underset{\sim}{b}$;*
4. *$\operatorname{rank}\left(\begin{bmatrix} \mathbf{A} & \underset{\sim}{b} \end{bmatrix}\right) = \operatorname{rank}(\mathbf{A})$, where $\begin{bmatrix} \mathbf{A} & \underset{\sim}{b} \end{bmatrix}$ is the $m \times (n+1)$ matrix whose last column is $\underset{\sim}{b}$.*

Proof

Proof. **1 and 2 are equivalent.** Statement 1 says $\underset{\sim}{b} = \mathbf{A}\underset{\sim}{x}$ for some $\underset{\sim}{x}$ (Definition 0.3), and $\mathcal{C}(\mathbf{A})$ is the set of vectors $\mathbf{A}\underset{\sim}{x}$ (Definition 1 in Column Space and Null Space), so $\underset{\sim}{b}$ is in it exactly when $\underset{\sim}{b} = \mathbf{A}\underset{\sim}{x}$ for some $\underset{\sim}{x}$.
1 implies 3. By Theorem 0.3, $\mathbf{G}\underset{\sim}{b}$ is a solution, so $\mathbf{A}(\mathbf{G}\underset{\sim}{b}) = \underset{\sim}{b}$, which is statement 3 (Theorem 1 in Matrix Multiplication).
3 implies 1. If $\mathbf{A}\mathbf{G}\underset{\sim}{b} = \underset{\sim}{b}$, then $\underset{\sim}{x} = \mathbf{G}\underset{\sim}{b}$ is a solution, since $\mathbf{A}(\mathbf{G}\underset{\sim}{b}) = (\mathbf{A}\mathbf{G})\underset{\sim}{b}$ (Theorem 1 in Matrix Multiplication).
2 and 4 are equivalent. Every column of $\mathbf{A}$ is a column of $\begin{bmatrix} \mathbf{A} & \underset{\sim}{b} \end{bmatrix}$, so $\mathcal{C}(\mathbf{A}) \subseteq \mathcal{C}\left(\begin{bmatrix} \mathbf{A} & \underset{\sim}{b} \end{bmatrix}\right)$.

- If $\underset{\sim}{b} \in \mathcal{C}(\mathbf{A})$, every combination of the columns of $\begin{bmatrix} \mathbf{A} & \underset{\sim}{b} \end{bmatrix}$ is a combination of the columns of $\mathbf{A}$ (Definition 1 in Column Space and Null Space), so the two column spaces (Definition 1 in Column Space and Null Space) coincide and their ranks agree (Theorem 1 in Rank-Nullity and Rank Factorization).
- If the ranks agree, the two column spaces have the same dimension (Theorem 1 in Rank-Nullity and Rank Factorization), so they coincide (Theorem 6 in Bases and Dimension), and $\underset{\sim}{b} \in \mathcal{C}\left(\begin{bmatrix} \mathbf{A} & \underset{\sim}{b} \end{bmatrix}\right) = \mathcal{C}(\mathbf{A})$.

□

Example 0.14 (Testing consistency with a generalized inverse)**.** For $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and the generalized inverse $\mathbf{G} = \begin{bmatrix} 1 & 5 \\ 7 & 0 \end{bmatrix}$, $\mathbf{AG} = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$.

- $\underset{\sim}{b} = (2, 0)$:

$$\begin{aligned} \mathbf{AG}\underset{\sim}{b} &= (2, 0) \\ &= \underset{\sim}{b}, \end{aligned}$$

  so the system is consistent.

- $\underset{\sim}{b} = (2, 1)$: $\mathbf{AG}\underset{\sim}{b} = (7, 0) \neq \underset{\sim}{b}$, so it is not, as Example 0.4 found directly.

Item 4 agrees. $\mathbf{A}$ has rank 1. For $\underset{\sim}{b} = (2, 0)$, $\begin{bmatrix} \mathbf{A} & \underset{\sim}{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ also has rank 1. For $\underset{\sim}{b} = (2, 1)$, $\begin{bmatrix} \mathbf{A} & \underset{\sim}{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ has rank 2, because $(1, 0)$ and $(2, 1)$ are linearly independent.

Theorem 0.11 (All solutions of a linear system)**.** *Let* $\mathbf{A}$ *be* $m \times n$ *with a generalized inverse* $\mathbf{G}$ *(Definition 0.1).*

1. $\mathcal{N}(\mathbf{A}) = \mathcal{C}(\mathbf{I}_n - \mathbf{GA})$*:* $\underset{\sim}{x}$ *solves* $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m$ *exactly when* $\underset{\sim}{x} = (\mathbf{I}_n - \mathbf{GA})\,\underset{\sim}{q}$ *for some* $q \in \mathbb{R}^n$.
2. *If* $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ *is consistent (Definition 0.3), then* $\underset{\sim}{x}$ *is a solution exactly when*

$$\underset{\sim}{x} = \mathbf{G}\underset{\sim}{b} + (\mathbf{I}_n - \mathbf{GA})\,\underset{\sim}{q} \quad \text{for some } \underset{\sim}{q} \in \mathbb{R}^n :$$

   one particular solution plus a vector of $\mathcal{N}(\mathbf{A})$.

Proof

Proof. **Part 1.** First,

$$\begin{aligned}
\mathbf{A}(\mathbf{I}_n - \mathbf{GA}) &= \mathbf{AI}_n - \mathbf{A}(\mathbf{GA}) && \text{(Theorem 0.1)}\\
&= \mathbf{AI}_n - \mathbf{AGA} && \text{(Theorem 1 in Matrix Multiplication)}\\
&= \mathbf{A} - \mathbf{AGA} && \text{(Theorem 4 in Matrix Multiplication)}\\
&= \mathbf{A} - \mathbf{A} && \text{(Definition 0.1)}\\
&= \mathbf{0}_{m\times n}, && \text{(arithmetic)}
\end{aligned}$$

so $\mathbf{A}(\mathbf{I}_n - \mathbf{GA})\,\underset{\sim}{q} = \underset{\sim}{0}_m$ for every $\underset{\sim}{q}$ (Theorem 1 in Matrix Multiplication), and $\mathcal{C}(\mathbf{I}_n - \mathbf{GA}) \subseteq \mathcal{N}(\mathbf{A})$. Conversely, if $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m$, then

$$\begin{aligned}
(\mathbf{I}_n - \mathbf{GA})\,\underset{\sim}{x} &= \mathbf{I}_n\underset{\sim}{x} - (\mathbf{GA})\,\underset{\sim}{x} && \text{(Theorem 0.1, with } \underset{\sim}{x} \text{ as the last factor)}\\
&= \underset{\sim}{x} - (\mathbf{GA})\,\underset{\sim}{x} && \text{(Theorem 4 in Matrix Multiplication)}\\
&= \underset{\sim}{x} - \mathbf{G}(\mathbf{A}\underset{\sim}{x}) && \text{(Theorem 1 in Matrix Multiplication)}\\
&= \underset{\sim}{x} - \mathbf{G}\,\underset{\sim}{0}_m && (\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m)\\
&= \underset{\sim}{x}, && (\mathbf{G}\,\underset{\sim}{0}_m = \underset{\sim}{0}_n)
\end{aligned}$$

so $\underset{\sim}{x}$ is in $\mathcal{C}(\mathbf{I}_n - \mathbf{GA})$, with $\underset{\sim}{q} = \underset{\sim}{x}$.

Part 2. $\mathbf{G}\underset{\sim}{b}$ is a solution (Theorem 0.3). For any $\underset{\sim}{x}$,

$$\begin{aligned}
\mathbf{A}(\underset{\sim}{x} - \mathbf{G}\underset{\sim}{b}) &= \mathbf{A}\underset{\sim}{x} - \mathbf{A}(\mathbf{G}\underset{\sim}{b}) && \text{(Theorem 4 in Subspaces and Rank, with coefficients 1 and } -1)\\
&= \mathbf{A}\underset{\sim}{x} - \mathbf{AG}\underset{\sim}{b} && \text{(Theorem 1 in Matrix Multiplication)}\\
&= \mathbf{A}\underset{\sim}{x} - \underset{\sim}{b}, && \text{(Theorem 0.10, statement 3)}
\end{aligned}$$

so $\underset{\sim}{x}$ solves $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ exactly when $\underset{\sim}{x} - \mathbf{G}\underset{\sim}{b} \in \mathcal{N}(\mathbf{A})$, that is, by part 1, exactly when $\underset{\sim}{x} - \mathbf{G}\underset{\sim}{b} = (\mathbf{I}_n - \mathbf{GA})\,\underset{\sim}{q}$ for some $\underset{\sim}{q}$. □

Example 0.15 (All solutions of a consistent system)**.** For $\mathbf{A} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $\mathbf{G} = \begin{bmatrix}1 & 5\\ 7 & 0\end{bmatrix}$ and $\underset{\sim}{b} = (2,0)$: $\mathbf{G}\underset{\sim}{b} = (2,14)$, $\mathbf{GA} = \begin{bmatrix}1 & 0\\ 7 & 0\end{bmatrix}$, and $\mathbf{I}_2 - \mathbf{GA} = \begin{bmatrix}0 & 0\\ -7 & 1\end{bmatrix}$, so

$$\underset{\sim}{x} = (2,14) + (0, -7q_1 + q_2), \qquad q_1, q_2 \in \mathbb{R}.$$

Since $-7q_1 + q_2$ can be any number, the solutions are exactly the vectors $(2,t)$, $t \in \mathbb{R}$, which is what

$$\begin{aligned}
\mathbf{A}(x_1, x_2) &= (x_1, 0)\\
&= (2,0)
\end{aligned}$$

says directly.

Corollary 0.2 (Solvable for every right-hand side, and solvable uniquely)**.** *Let* $\mathbf{A}$ *be* $m \times n$*.*

1. $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ *is consistent for every* $\underset{\sim}{b} \in \mathbb{R}^m$ *exactly when* $\text{rank}(\mathbf{A}) = m$*.*
2. *A consistent system* $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ *has exactly one solution exactly when* $\text{rank}(\mathbf{A}) = n$*.*

Proof

Proof. **Part 1.** By Theorem 0.10, the system is consistent for every $\underset{\sim}{b}$ exactly when $\mathcal{C}(\mathbf{A}) = \mathbb{R}^m$. $\mathcal{C}(\mathbf{A})$ is a subspace of $\mathbb{R}^m$ (Theorem 1 in Column Space and Null Space) with dimension $\text{rank}(\mathbf{A})$ (Theorem 1 in Rank-Nullity and Rank Factorization), and $\dim(\mathbb{R}^m) = m$ (Example 6 in Bases and Dimension). If the dimension is m, then $\mathcal{C}(\mathbf{A}) = \mathbb{R}^m$ (Theorem 6 in Bases and Dimension); if $\mathcal{C}(\mathbf{A}) = \mathbb{R}^m$, the dimension is m.
Part 2. By Theorem 0.11, the solutions are a particular solution plus the vectors of $\mathcal{N}(\mathbf{A})$, so there is exactly one solution exactly when $\mathcal{N}(\mathbf{A}) = \{\underset{\sim}{0}_n\}$, that is, when $\text{nullity}(\mathbf{A}) = 0$ (a subspace of dimension 0 has the empty list as a basis, whose span is $\{\underset{\sim}{0}\}$, by Definition 4 in Bases and Dimension and Definition 5 in Subspaces and Rank; and $\{\underset{\sim}{0}\}$ has dimension 0, by Example 6 in Bases and Dimension). By Theorem 2 in Rank-Nullity and Rank Factorization, that is when $\text{rank}(\mathbf{A}) = n$. □

Example 0.16 (Three systems)**.**

- The 2×3 matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ of Example 6 in Rank-Nullity and Rank Factorization has rank $2 = m$, so every system with it is consistent, but $2 < 3 = n$, so the solutions are never unique.
- The 3×2 matrix $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ of Example 1 in Design Matrices and Projections has rank $2 = n$, so a consistent system has one solution, but $2 < 3 = m$, so some systems are inconsistent: $(1, 0, 0) \notin \mathcal{C}(\mathbf{X})$, because $a\,(1, 1, 1) + b\,(1, 2, 3)$ has equal differences between consecutive entries, and $(1, 0, 0)$ does not.
- An invertible $p \times p$ matrix $\mathbf{A}$ has rank p: if $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}$ then

$$\begin{aligned} \underset{\sim}{x} &= \mathbf{A}^{-1}\mathbf{A}\underset{\sim}{x} \\ &= \underset{\sim}{0}, \end{aligned}$$

  so its nullity is 0 and its rank is p (Theorem 2 in Rank-Nullity and Rank Factorization). So every system with it is consistent (part 1, with $p = m$) and has exactly one solution (part 2, with $p = n$), namely $\mathbf{A}^{-1}\underset{\sim}{b}$, since $\mathbf{A}\,(\mathbf{A}^{-1}\underset{\sim}{b}) = \underset{\sim}{b}$.

Theorem 0.12 (The projection onto a column space, with any generalized inverse)**.** *Let* $\mathbf{X}$ *be an* $n \times p$ *matrix of any rank, and let* $\mathbf{G}$ *be any generalized inverse of* $\mathbf{X}^\top\mathbf{X}$ *(Definition 0.1). Then for every* $y \in \mathbb{R}^n$, $\mathbf{X}\mathbf{G}\mathbf{X}^\top y$ *is the orthogonal projection of* y *onto* $\mathcal{C}(\mathbf{X})$ *(Definition 3 in Design Matrices and Projections). So the matrix* $\mathbf{X}\mathbf{G}\mathbf{X}^\top$ *is the same for every choice of* $\mathbf{G}$*, and when* $\text{rank}(\mathbf{X}) = p$ *it is the hat matrix* $\mathbf{H}$ *(Definition 2 in Design Matrices and Projections).*

Proof

Proof. **$\mathbf{X}^\top y$ is in $\mathcal{C}(\mathbf{X}^\top\mathbf{X})$.** $\mathcal{C}(\mathbf{X}^\top\mathbf{X}) \subseteq \mathcal{C}(\mathbf{X}^\top)$, since $(\mathbf{X}^\top\mathbf{X})\underset{\sim}{z} = \mathbf{X}^\top(\mathbf{X}\underset{\sim}{z})$ (Theorem 1 in Matrix Multiplication). Both are subspaces of $\mathbb{R}^p$ (Theorem 1 in Column Space and Null Space) with the same dimension: $\text{rank}(\mathbf{X}^\top\mathbf{X}) = \text{rank}(\mathbf{X}^\top)$ (Theorem 4 in Rank-Nullity and Rank Factorization, Theorem 1 in Rank-Nullity and Rank Factorization). So $\mathcal{C}(\mathbf{X}^\top\mathbf{X}) = \mathcal{C}(\mathbf{X}^\top)$ (Theorem 6 in Bases and Dimension), and since $\mathbf{X}^\top y \in \mathcal{C}(\mathbf{X}^\top)$ (Definition 1 in Column Space and Null Space), $\mathbf{X}^\top\underset{\sim}{y} = \mathbf{X}^\top\mathbf{X}\underset{\sim}{z}$ for some $\underset{\sim}{z} \in \mathbb{R}^p$.
The split. Let

$$\begin{aligned}\underset{\sim}{u} &= \mathbf{X}\mathbf{G}\mathbf{X}^\top y \\ &= \mathbf{X}\left(\mathbf{G}\mathbf{X}^\top\underset{\sim}{y}\right)\end{aligned}$$

(Theorem 1 in Matrix Multiplication), which is in $\mathcal{C}(\mathbf{X})$ (Definition 1 in Column Space and Null Space). For $\underset{\sim}{y} - \underset{\sim}{u}$,

$$\begin{aligned}
\mathbf{X}^\top(\underset{\sim}{y} - \underset{\sim}{u}) &= \mathbf{X}^\top\underset{\sim}{y} - \mathbf{X}^\top\underset{\sim}{u} && \text{(Theorem 4 in Subspaces and Rank, with coefficients 1 and } -1\text{)} \\
&= \mathbf{X}^\top\underset{\sim}{y} - \mathbf{X}^\top(\mathbf{X}\mathbf{G}\mathbf{X}^\top\underset{\sim}{y}) && \text{(substitute } \underset{\sim}{u}\text{)} \\
&= \mathbf{X}^\top\underset{\sim}{y} - (\mathbf{X}^\top\mathbf{X})\,\mathbf{G}\,(\mathbf{X}^\top\underset{\sim}{y}) && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{X}^\top\underset{\sim}{y} - (\mathbf{X}^\top\mathbf{X})\,\mathbf{G}\,(\mathbf{X}^\top\mathbf{X})\,\underset{\sim}{z} && \text{(first step)} \\
&= \mathbf{X}^\top\underset{\sim}{y} - ((\mathbf{X}^\top\mathbf{X})\,\mathbf{G}\,(\mathbf{X}^\top\mathbf{X}))\,\underset{\sim}{z} && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{X}^\top\underset{\sim}{y} - (\mathbf{X}^\top\mathbf{X})\,\underset{\sim}{z} && \text{(Definition 0.1)} \\
&= \mathbf{X}^\top\underset{\sim}{y} - \mathbf{X}^\top\underset{\sim}{y} && \text{(first step)} \\
&= \underset{\sim}{0}_p, && \text{(arithmetic)}
\end{aligned}$$

so $y - \underset{\sim}{u} \in \mathcal{N}(\mathbf{X}^\top) = \mathcal{C}(\mathbf{X})^\perp$ (Theorem 3 in Orthogonal Complements and the Fundamental Theorem). By the uniqueness in Theorem 4 in Orthogonal Complements and the Fundamental Theorem, $\underset{\sim}{u}$ is the orthogonal projection of y onto $\mathcal{C}(\mathbf{X})$.
Same matrix. Every choice of $\mathbf{G}$ gives a matrix that projects every y onto $\mathcal{C}(\mathbf{X})$, so all choices give the same matrix (Theorem 5 in Design Matrices and Projections, part 2). When $\text{rank}(\mathbf{X}) = p$, $(\mathbf{X}^\top\mathbf{X})^{-1}$ is one choice (Theorem 1 in Design Matrices and Projections, Theorem 0.2), and it gives $\mathbf{H}$. □

Example 0.17 (A design matrix with a repeated column)**.** Let $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, an intercept column entered twice, so $\operatorname{rank}(\mathbf{X}) = 1 < 2$ and $\mathbf{X}^\top\mathbf{X} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ is not invertible: its rank is $\operatorname{rank}(\mathbf{X}) = 1 < 2$ (Theorem 4 in Rank-Nullity and Rank Factorization), while an invertible 2×2 matrix has rank 2 (Example 0.16). The matrix $\mathbf{G} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$ is a generalized inverse of it:

$$
\begin{aligned}
(\mathbf{X}^\top\mathbf{X})\,\mathbf{G}\,(\mathbf{X}^\top\mathbf{X}) &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} && \text{(Definition 1 in Matrix Multiplication, for the first product)} \\
&= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}. && \text{(Definition 1 in Matrix Multiplication)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{X}\mathbf{G}\mathbf{X}^\top &= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} && \text{(Definition 1 in Matrix Multiplication, for } \mathbf{X}\mathbf{G}\text{)} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, && \text{(Definition 1 in Matrix Multiplication)}
\end{aligned}
$$

the hat matrix of the intercept-only model (Example 2 in Design Matrices and Projections). Another generalized inverse, $\mathbf{G}' = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, which is one because $(\mathbf{X}^\top\mathbf{X})\,\mathbf{G}' = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and that times $\mathbf{X}^\top\mathbf{X}$ is again $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, gives $\mathbf{X}\mathbf{G}' = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$ and the same $\mathbf{X}\mathbf{G}'\mathbf{X}^\top = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, as Theorem 0.12 says it must. Both project onto $\mathcal{C}(\mathbf{X}) = \operatorname{span}\{(1, 1)\}$, the column space of the intercept-only design.

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