

Direct Sums and Orthogonal Complements

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Sums and direct sums

Source

This section, Section and Section are adapted from Zhou [1], used under the MIT License (see the license text in [Section 2 in Subspaces and Rank](#)). The source proves that a subspace and its orthogonal complement together make up $\mathbb{R}^p$ by extending an orthonormal basis; the proof here uses the rank-nullity theorem ([Theorem 7 in Column Space, Null Space and Rank-Nullity](#)) instead. The source defines the orthogonal complement inside an ambient vector space $V \subseteq \mathbb{R}^m$; here the ambient space is always $\mathbb{R}^p$, which is all that the fundamental theorem and the projections need.

Definition 0.1 (Sum of subspaces). The **sum** of two subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbb{R}^p$ ([Definition 4 in Subspaces and Rank](#)) is the set of all sums of a vector from $\mathcal{S}_1$ and a vector from $\mathcal{S}_2$:

$$\mathcal{S}_1 + \mathcal{S}_2 \stackrel{\text{def}}{=} \{\tilde{x}_1 + \tilde{x}_2 : \tilde{x}_1 \in \mathcal{S}_1, \tilde{x}_2 \in \mathcal{S}_2\}.$$

Example 0.1 (Two lines sum to a plane). Let $\mathcal{S}_1 = \text{span}\{(1, 0, 0)\}$ and $\mathcal{S}_2 = \text{span}\{(0, 1, 0)\}$, two lines in $\mathbb{R}^3$. A vector of $\mathcal{S}_1 + \mathcal{S}_2$ has the form $a(1, 0, 0) + b(0, 1, 0) = (a, b, 0)$, and a and b can be any numbers, so $\mathcal{S}_1 + \mathcal{S}_2 = \{(a, b, 0) : a, b \in \mathbb{R}\}$, the plane $z = 0$.

Example 0.2 (The sum is not the union). For the two lines in [Example 0.1](#), the union $\mathcal{S}_1 \cup \mathcal{S}_2$ contains $(1, 0, 0)$ and $(0, 1, 0)$ but not their sum $(1, 1, 0)$: every multiple of $(1, 0, 0)$ has second entry 0, and every multiple of $(0, 1, 0)$ has first entry 0. So the union is not closed under addition and is not a subspace, while the sum, the whole plane $z = 0$, is.

Theorem 0.1 (Sums and intersections of subspaces are subspaces). *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subspaces of $\mathbb{R}^p$ (Definition 4 in Subspaces and Rank), then $\mathcal{S}_1 + \mathcal{S}_2$ (Definition 0.1) and $\mathcal{S}_1 \cap \mathcal{S}_2$ are subspaces of $\mathbb{R}^p$.*

i Proof

Proof. Both $\mathcal{S}_1$ and $\mathcal{S}_2$ contain $\tilde{0}$ (Theorem 1 in Subspaces and Rank).

The sum. It is not empty: it contains $\tilde{0} + \tilde{0} = \tilde{0}$. Take two of its vectors, $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$ and $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$, with $\tilde{u}_1, \tilde{w}_1 \in \mathcal{S}_1$ and $\tilde{u}_2, \tilde{w}_2 \in \mathcal{S}_2$, and a number c . Then

$$\begin{aligned}\tilde{u} + \tilde{w} &= (\tilde{u}_1 + \tilde{u}_2) + (\tilde{w}_1 + \tilde{w}_2) \quad (\text{substitute}) \\ &= (\tilde{u}_1 + \tilde{w}_1) + (\tilde{u}_2 + \tilde{w}_2), \quad (\text{regroup; vector addition is entrywise})\end{aligned}$$

where $\tilde{u}_1 + \tilde{w}_1 \in \mathcal{S}_1$ and $\tilde{u}_2 + \tilde{w}_2 \in \mathcal{S}_2$, because each subspace is closed under addition. Likewise

$$\begin{aligned}c\tilde{u} &= c(\tilde{u}_1 + \tilde{u}_2) \quad (\text{substitute}) \\ &= c\tilde{u}_1 + c\tilde{u}_2, \quad (\text{distribute } c \text{ entrywise})\end{aligned}$$

where $c\tilde{u}_1 \in \mathcal{S}_1$ and $c\tilde{u}_2 \in \mathcal{S}_2$, because each subspace is closed under scalar multiplication. So $\tilde{u} + \tilde{w}$ and $c\tilde{u}$ are in $\mathcal{S}_1 + \mathcal{S}_2$.

The intersection. It is not empty: it contains $\tilde{0}$. If $\tilde{u}$ and $\tilde{w}$ are in both $\mathcal{S}_1$ and $\mathcal{S}_2$, then $\tilde{u} + \tilde{w}$ and $c\tilde{u}$ are in $\mathcal{S}_1$, because $\mathcal{S}_1$ is a subspace, and in $\mathcal{S}_2$ for the same reason, so they are in $\mathcal{S}_1 \cap \mathcal{S}_2$. $\square$

Example 0.3 (Two planes in $\mathbb{R}^3$). Let $\mathcal{S}_1 = \{(a, b, 0) : a, b \in \mathbb{R}\}$, the plane $z = 0$, and $\mathcal{S}_2 = \{(0, b, c) : b, c \in \mathbb{R}\}$, the plane $x = 0$.

- **Intersection:** a vector in both has third entry 0 and first entry 0, so $\mathcal{S}_1 \cap \mathcal{S}_2 = \{(0, b, 0) : b \in \mathbb{R}\}$, the y -axis, which is a line through the origin.
- **Sum:** $(a, b, 0) + (0, b', c) = (a, b + b', c)$, and any (x, y, z) arises this way, with $a = x$, $b = y$, $b' = 0$ and $c = z$. So $\mathcal{S}_1 + \mathcal{S}_2 = \mathbb{R}^3$.

Theorem 0.2 (The dimension of a sum of subspaces). *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subspaces of $\mathbb{R}^p$, then*

$$\dim(\mathcal{S}_1 + \mathcal{S}_2) = \dim(\mathcal{S}_1) + \dim(\mathcal{S}_2) - \dim(\mathcal{S}_1 \cap \mathcal{S}_2).$$

i Proof

Proof. All three sets in the formula are subspaces (Theorem 0.1), so each has a dimension (Definition 9 in Subspaces and Rank). Let $k = \dim(\mathcal{S}_1 \cap \mathcal{S}_2)$, $d_1 = \dim(\mathcal{S}_1)$ and $d_2 = \dim(\mathcal{S}_2)$.

Build a list. Choose a basis $\tilde{z}_1, \dots, \tilde{z}_k$ of $\mathcal{S}_1 \cap \mathcal{S}_2$ (Theorem 9 in Subspaces and Rank). These vectors are linearly independent and lie in $\mathcal{S}_1$, so Theorem 9 in Subspaces and Rank extends them to a basis $\tilde{z}_1, \dots, \tilde{z}_k, \tilde{x}_1, \dots, \tilde{x}_{d_1-k}$ of $\mathcal{S}_1$, which has d_1 vectors (Theorem 7 in Subspaces and Rank). In the same way, extend them to a basis $\tilde{z}_1, \dots, \tilde{z}_k, \tilde{y}_1, \dots, \tilde{y}_{d_2-k}$ of $\mathcal{S}_2$. We show that the $\tilde{z}$'s, $\tilde{x}$'s and $\tilde{y}$'s together, $k + (d_1 - k) + (d_2 - k) = d_1 + d_2 - k$ vectors, are a basis of $\mathcal{S}_1 + \mathcal{S}_2$ (Definition 6 in Subspaces and Rank).

They span $\mathcal{S}_1 + \mathcal{S}_2$. Any $\tilde{v} \in \mathcal{S}_1 + \mathcal{S}_2$ is $\tilde{v}_1 + \tilde{v}_2$ with $\tilde{v}_1 \in \mathcal{S}_1$ and $\tilde{v}_2 \in \mathcal{S}_2$. $\tilde{v}_1$ is a linear combination of the $\tilde{z}$'s and $\tilde{x}$'s, and $\tilde{v}_2$ is a linear combination of the $\tilde{z}$'s and $\tilde{y}$'s, so their sum is a linear combination of all three groups. Conversely, each vector in the list is in $\mathcal{S}_1 + \mathcal{S}_2$ (for example, $\tilde{x}_1 = \tilde{x}_1 + \tilde{0}$), and $\mathcal{S}_1 + \mathcal{S}_2$ is a subspace, so it contains every linear combination of them.

They are linearly independent. Suppose

$$\sum_{i=1}^k a_i \tilde{z}_i + \sum_{j=1}^{d_1-k} b_j \tilde{x}_j + \sum_{l=1}^{d_2-k} c_l \tilde{y}_l = \tilde{0}.$$

Let $\tilde{w} \stackrel{\text{def}}{=} \sum_l c_l \tilde{y}_l$. Then $\tilde{w} \in \mathcal{S}_2$, because it is a linear combination of vectors of $\mathcal{S}_2$. Subtracting $\sum_i a_i \tilde{z}_i + \sum_j b_j \tilde{x}_j$ from both sides of the supposed equation gives

$$\tilde{w} = \sum_{i=1}^k (-a_i) \tilde{z}_i + \sum_{j=1}^{d_1-k} (-b_j) \tilde{x}_j,$$

a linear combination of vectors of $\mathcal{S}_1$, so $\tilde{w} \in \mathcal{S}_1$ too. So $\tilde{w} \in \mathcal{S}_1 \cap \mathcal{S}_2$, and $\tilde{w} = \sum_i e_i \tilde{z}_i$ for some numbers e_i , since the $\tilde{z}$'s span the intersection. Then

$$\begin{aligned} \tilde{0} &= \tilde{w} - \tilde{w} && \text{(a vector minus itself)} \\ &= \sum_{i=1}^k e_i \tilde{z}_i - \sum_{l=1}^{d_2-k} c_l \tilde{y}_l && \text{(substitute the two expressions for } \tilde{w}) \\ &= \sum_{i=1}^k e_i \tilde{z}_i + \sum_{l=1}^{d_2-k} (-c_l) \tilde{y}_l, && \text{(write each subtraction as adding } -1 \text{ times)} \end{aligned}$$

and because the $\tilde{z}$'s and $\tilde{y}$'s are a basis of $\mathcal{S}_2$, every $c_l = 0$. The supposed equation then reads $\sum_i a_i \tilde{z}_i + \sum_j b_j \tilde{x}_j = \tilde{0}$, and because the $\tilde{z}$'s and $\tilde{x}$'s are a basis of $\mathcal{S}_1$, every a_i and b_j is 0.

So $\dim(\mathcal{S}_1 + \mathcal{S}_2) = d_1 + d_2 - k$. When some of the groups are empty (for example $k = 0$, or $\mathcal{S}_1 = \mathcal{S}_1 \cap \mathcal{S}_2$), the sums over them are 0 and the argument goes through unchanged. $\square$

Example 0.4 (Two planes in $\mathbb{R}^3$, by dimension). For the planes $z = 0$ and $x = 0$ in Example 0.3, each has dimension 2: the plane $z = 0$ has basis $(1, 0, 0), (0, 1, 0)$ (Example 16 in Subspaces and Rank), and the plane $x = 0$ has basis $(0, 1, 0), (0, 0, 1)$ by the same argument. Their intersection, the y -axis $\text{span}\{(0, 1, 0)\}$, has dimension 1, because the single nonzero vector $(0, 1, 0)$ is linearly independent ($c(0, 1, 0) = (0, c, 0)$ is $\underline{0}$ only if $c = 0$) and so is a basis of it. Theorem 0.2 gives

$$\begin{aligned}\dim(\mathcal{S}_1 + \mathcal{S}_2) &= 2 + 2 - 1 \\ &= 3,\end{aligned}$$

which agrees with $\mathcal{S}_1 + \mathcal{S}_2 = \mathbb{R}^3$ (Example 0.3) and $\dim(\mathbb{R}^3) = 3$ (Example 16 in Subspaces and Rank).

Corollary 0.1 (The dimension of a sum is at most the sum of the dimensions). *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subspaces of $\mathbb{R}^p$, then*

$$\dim(\mathcal{S}_1 + \mathcal{S}_2) \leq \dim(\mathcal{S}_1) + \dim(\mathcal{S}_2),$$

with equality exactly when $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\underline{0}\}$.

i Proof

Proof. By Theorem 0.2, the two sides differ by $\dim(\mathcal{S}_1 \cap \mathcal{S}_2) \geq 0$. That dimension is 0 exactly when a basis of the intersection is the empty list, that is, when the intersection is the span of the empty list, $\{\underline{0}\}$ (Definition 5 in Subspaces and Rank, Example 16 in Subspaces and Rank). $\square$

Example 0.5 (Equality and strict inequality).

- For the lines $\mathcal{S}_1 = \text{span}\{(1, 0, 0)\}$ and $\mathcal{S}_2 = \text{span}\{(0, 1, 0)\}$ in Example 0.1, a common vector satisfies $a(1, 0, 0) = b(0, 1, 0)$, so $(a, -b, 0) = \underline{0}$ and

$$\begin{aligned}a &= b \\ &= 0.\end{aligned}$$

The intersection is $\{\underline{0}\}$, and indeed

$$\begin{aligned}\dim(\mathcal{S}_1 + \mathcal{S}_2) &= 2 \\ &= 1 + 1:\end{aligned}$$

the sum is the plane $z = 0$ (Example 0.1), which has dimension 2 (Example 16 in

Subspaces and Rank), and each line has dimension 1, with its spanning vector as a basis.

- For

$$\begin{aligned}\mathcal{S}_1 &= \mathcal{S}_2 \\ &= \text{span}\{(1, 0, 0)\},\end{aligned}$$

the sum is the same line, because $a(1, 0, 0) + b(1, 0, 0) = (a+b)(1, 0, 0)$, so $\dim(\mathcal{S}_1 + \mathcal{S}_2) = 1 < 1 + 1$, a [strict inequality](#).

Theorem 0.3 (Rank is subadditive). *For $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$,*

$$\text{rank}(\mathbf{A} + \mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}).$$

i Proof

Proof. $\mathcal{C}(\mathbf{A} + \mathbf{B}) \subseteq \mathcal{C}(\mathbf{A}) + \mathcal{C}(\mathbf{B})$. Any vector of $\mathcal{C}(\mathbf{A} + \mathbf{B})$ is $(\mathbf{A} + \mathbf{B})\tilde{x}$ for some $\tilde{x} \in \mathbb{R}^n$ ([Definition 1 in Column Space, Null Space and Rank-Nullity](#)), and $(\mathbf{A} + \mathbf{B})\tilde{x} = \mathbf{A}\tilde{x} + \mathbf{B}\tilde{x}$ ([Theorem 6 in Matrices](#), with $\tilde{x}$ as an $n \times 1$ matrix), a vector of $\mathcal{C}(\mathbf{A})$ plus a vector of $\mathcal{C}(\mathbf{B})$.

Compare dimensions. All the sets involved are subspaces of $\mathbb{R}^m$ ([Theorem 1 in Column Space, Null Space and Rank-Nullity](#), [Theorem 0.1](#)), so

$$\begin{aligned}\text{rank}(\mathbf{A} + \mathbf{B}) &= \dim(\mathcal{C}(\mathbf{A} + \mathbf{B})) && \text{(Theorem 6 in Column Space, Null Space and Rank-Nullity)} \\ &\leq \dim(\mathcal{C}(\mathbf{A}) + \mathcal{C}(\mathbf{B})) && \text{(the containment, and part 2 of Theorem 10 in Subspaces and Rank)} \\ &\leq \dim(\mathcal{C}(\mathbf{A})) + \dim(\mathcal{C}(\mathbf{B})) && \text{(Corollary 0.1)} \\ &= \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}). && \text{(Theorem 6 in Column Space, Null Space and Rank-Nullity)}\end{aligned}$$

□

Example 0.6 (Subadditivity with equality and without). Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Each has one nonzero column and one zero column. The nonzero column on its own is linearly independent ($c\tilde{v} = \tilde{0}_2$ with $\tilde{v} \neq \tilde{0}_2$ forces $c = 0$), and the two columns together are not, since 1 times the zero column is $\tilde{0}_2$; so each matrix has rank 1 ([Definition 2 in Subspaces and Rank](#)). By the same argument, $-\mathbf{A}$ has rank 1.

- $\mathbf{A} + \mathbf{B}$ is the 2×2 identity matrix, which has rank 2 ([Example 12 in Column Space,](#)

Null Space and Rank-Nullity), so the bound $2 \leq 1 + 1$ holds with equality.

- $\mathbf{A} + (-\mathbf{A})$ is the 2×2 zero matrix, which has rank 0: any nonempty list of its columns contains $\mathbf{0}_2$, and $1 \cdot \mathbf{0}_2 = \mathbf{0}_2$. So the bound $0 \leq \text{rank}(\mathbf{A}) + \text{rank}(-\mathbf{A}) = 1 + 1$ is strict.

Definition 0.2 (Complementary subspaces). Let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{V}$ be subspaces of $\mathbb{R}^p$. $\mathcal{S}_1$ and $\mathcal{S}_2$ are **complementary** in $\mathcal{V}$ if

$$\mathcal{V} = \mathcal{S}_1 + \mathcal{S}_2 \quad \text{and} \quad \mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$$

(Definition 0.1).

Definition 0.3 (Direct sum). If $\mathcal{S}_1$ and $\mathcal{S}_2$ are complementary in $\mathcal{V}$ (Definition 0.2), then $\mathcal{V}$ is the **direct sum** [2] of $\mathcal{S}_1$ and $\mathcal{S}_2$, written $\mathcal{V} = \mathcal{S}_1 \oplus \mathcal{S}_2$.

Example 0.7 ($\mathbb{R}^3$ is a plane plus a line). Let $\mathcal{S}_1 = \{(a, b, 0) : a, b \in \mathbb{R}\}$, the plane $z = 0$, and $\mathcal{S}_2 = \{(0, 0, c) : c \in \mathbb{R}\}$, the z -axis.

- **Sum:** any $(x, y, z) \in \mathbb{R}^3$ is $(x, y, 0) + (0, 0, z)$, so $\mathcal{S}_1 + \mathcal{S}_2 = \mathbb{R}^3$.
- **Intersection:** a vector in both has third entry 0 and first and second entries 0, so $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$.

So $\mathbb{R}^3 = \mathcal{S}_1 \oplus \mathcal{S}_2$.

Example 0.8 (Two planes that sum to $\mathbb{R}^3$ but not directly). The planes $z = 0$ and $x = 0$ sum to $\mathbb{R}^3$ (Example 0.3), but they share the y -axis, so they are not complementary, and $\mathbb{R}^3$ is not their direct sum. The shared line shows up as more than one way to split a vector:

$$\begin{aligned} (0, 1, 0) &= (0, 1, 0) + \mathbf{0} \\ &= \mathbf{0} + (0, 1, 0), \end{aligned}$$

with the first term in the plane $z = 0$ and the second in the plane $x = 0$ both times.

Theorem 0.4 (Three ways to recognize a direct sum). Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subspaces of $\mathbb{R}^p$, and let $\mathcal{V} = \mathcal{S}_1 + \mathcal{S}_2$. The following statements are equivalent:

1. $\mathcal{V} = \mathcal{S}_1 \oplus \mathcal{S}_2$ (Definition 0.3).
2. $\dim(\mathcal{V}) = \dim(\mathcal{S}_1) + \dim(\mathcal{S}_2)$.
3. Every $\mathbf{x} \in \mathcal{V}$ can be written as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with $\mathbf{x}_1 \in \mathcal{S}_1$ and $\mathbf{x}_2 \in \mathcal{S}_2$ in only one

way.

i Proof

Proof. Since $\mathcal{V} = \mathcal{S}_1 + \mathcal{S}_2$ is given, statement 1 says exactly that $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$.

1 and 2 are equivalent. By Corollary 0.1, $\dim(\mathcal{S}_1 + \mathcal{S}_2) = \dim(\mathcal{S}_1) + \dim(\mathcal{S}_2)$ exactly when $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$.

1 implies 3. Every $\tilde{x} \in \mathcal{V}$ has at least one such expression, because $\mathcal{V} = \mathcal{S}_1 + \mathcal{S}_2$. Suppose

$$\begin{aligned}\tilde{x} &= \tilde{u}_1 + \tilde{u}_2 \\ &= \tilde{v}_1 + \tilde{v}_2,\end{aligned}$$

with $\tilde{u}_1, \tilde{v}_1 \in \mathcal{S}_1$ and $\tilde{u}_2, \tilde{v}_2 \in \mathcal{S}_2$. Then

$$\begin{aligned}\tilde{u}_1 - \tilde{v}_1 &= (\tilde{u}_1 + \tilde{u}_2) - \tilde{u}_2 - \tilde{v}_1 && \text{(add and subtract } \tilde{u}_2\text{)} \\ &= (\tilde{v}_1 + \tilde{v}_2) - \tilde{u}_2 - \tilde{v}_1 && \text{(substitute } \tilde{u}_1 + \tilde{u}_2 = \tilde{v}_1 + \tilde{v}_2\text{)} \\ &= \tilde{v}_2 - \tilde{u}_2. && \text{(cancel } \tilde{v}_1\text{)}\end{aligned}$$

The left side is in $\mathcal{S}_1$ and the right side is in $\mathcal{S}_2$, because each subspace is closed under addition and scalar multiplication (the difference is $\tilde{u}_1 + (-1)\tilde{v}_1$). So both sides are in $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$, which gives $\tilde{u}_1 = \tilde{v}_1$ and $\tilde{u}_2 = \tilde{v}_2$.

3 implies 1. Take any $\tilde{x} \in \mathcal{S}_1 \cap \mathcal{S}_2$. Then $\tilde{x} = \tilde{x} + \mathbf{0}$, with $\tilde{x} \in \mathcal{S}_1$ and $\mathbf{0} \in \mathcal{S}_2$, and $\tilde{x} = \mathbf{0} + \tilde{x}$, with $\mathbf{0} \in \mathcal{S}_1$ and $\tilde{x} \in \mathcal{S}_2$ (Theorem 1 in Subspaces and Rank). By statement 3 these two expressions are the same, so $\tilde{x} = \mathbf{0}$, and $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$. $\square$

Example 0.9 (Checking the three statements). For the plane $z = 0$ and the z -axis in Example 0.7:

- the dimensions add up:

$$\begin{aligned}2 + 1 &= 3 \\ &= \dim(\mathbb{R}^3),\end{aligned}$$

since the plane has dimension 2 (Example 16 in Subspaces and Rank), the z -axis has the single nonzero vector $(0, 0, 1)$ as a basis, and $\dim(\mathbb{R}^3) = 3$ (Example 16 in Subspaces and Rank);

- the split is unique: if

$$\begin{aligned}(x, y, z) &= (a, b, 0) + (0, 0, c) \\ &= (a, b, c),\end{aligned}$$

then $a = x$, $b = y$ and $c = z$.

For the planes $z = 0$ and $x = 0$ in Example 0.8, both statements fail:

$$\begin{aligned} 2 + 2 &= 4 \\ &\neq 3 \\ &= \dim(\mathcal{S}_1 + \mathcal{S}_2) \end{aligned}$$

(Example 0.4), and $(0, 1, 0)$ splits in two ways.

Definition 0.4 (Projection onto a subspace along a complement). Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathbb{R}^p$ with $\mathbb{R}^p = \mathcal{S} \oplus \mathcal{T}$ (Definition 0.3), and let $y \in \mathbb{R}^p$. By Theorem 0.4, $y = \tilde{u} + \tilde{v}$ for exactly one $\tilde{u} \in \mathcal{S}$ and $\tilde{v} \in \mathcal{T}$. The vector $\tilde{u}$ is the **projection** [3] of y onto $\mathcal{S}$ along $\mathcal{T}$ [4], Definition 6.3, p. 163.

Example 0.10 (The projection depends on the complement). Let $\mathcal{S} = \text{span}\{(1, 0)\}$, the horizontal axis in $\mathbb{R}^2$, and let $y = (3, 2)$.

- **Along $\mathcal{T}_1 = \text{span}\{(1, -1)\}$:** $(3, 2) = (5, 0) + (-2, 2)$, with $(5, 0) \in \mathcal{S}$ and $(-2, 2) = -2 \cdot (1, -1) \in \mathcal{T}_1$, so the projection of $(3, 2)$ onto $\mathcal{S}$ along $\mathcal{T}_1$ is $(5, 0)$.
- **Along $\mathcal{T}_2 = \text{span}\{(0, 1)\}$:** $(3, 2) = (3, 0) + (0, 2)$, with $(3, 0) \in \mathcal{S}$ and $(0, 2) \in \mathcal{T}_2$, so the projection of $(3, 2)$ onto $\mathcal{S}$ along $\mathcal{T}_2$ is $(3, 0)$.

Each of $\mathcal{T}_1$ and $\mathcal{T}_2$ is a line that meets $\mathcal{S}$ only at 0 , so

$$\begin{aligned} \mathbb{R}^2 &= \mathcal{S} \oplus \mathcal{T}_1 \\ &= \mathcal{S} \oplus \mathcal{T}_2 \end{aligned}$$

(Definition 0.3, Theorem 0.4). The same vector has a different projection onto the same subspace when the complement changes.

Orthogonal complements

Theorem 0.5 (The dot product is linear in each slot). For vectors $\tilde{x}, \tilde{u}, \tilde{w} \in \mathbb{R}^p$ and numbers a, b ,

$$\tilde{x} \cdot (a \tilde{u} + b \tilde{w}) = a (\tilde{x} \cdot \tilde{u}) + b (\tilde{x} \cdot \tilde{w})$$

and

$$(a \tilde{u} + b \tilde{w}) \cdot \tilde{x} = a (\tilde{u} \cdot \tilde{x}) + b (\tilde{w} \cdot \tilde{x}).$$

i Proof

Proof. **Second slot.**

$$\begin{aligned}
 \underline{x} \cdot (a \underline{u} + b \underline{w}) &= \sum_{i=1}^p x_i (a \underline{u} + b \underline{w})_i && \text{(Definition 8 in Vectors)} \\
 &= \sum_{i=1}^p x_i (a u_i + b w_i) && \text{(Definition 6 in Matrices, Definition 7 in Vectors)} \\
 &= \sum_{i=1}^p (x_i a u_i + x_i b w_i) && \text{(distribute each } x_i) \\
 &= \sum_{i=1}^p (a x_i u_i + b x_i w_i) && \text{(commute the factors in each product)} \\
 &= \sum_{i=1}^p a x_i u_i + \sum_{i=1}^p b x_i w_i && \text{(split the finite sum)} \\
 &= a \sum_{i=1}^p x_i u_i + b \sum_{i=1}^p x_i w_i && \text{(factor } a \text{ and } b \text{ out of the sums)} \\
 &= a (\underline{x} \cdot \underline{u}) + b (\underline{x} \cdot \underline{w}). && \text{(Definition 8 in Vectors)}
 \end{aligned}$$

First slot.

$$\begin{aligned}
 (a \underline{u} + b \underline{w}) \cdot \underline{x} &= \underline{x} \cdot (a \underline{u} + b \underline{w}) && \text{(Theorem 1 in Vectors)} \\
 &= a (\underline{x} \cdot \underline{u}) + b (\underline{x} \cdot \underline{w}) && \text{(second slot)} \\
 &= a (\underline{u} \cdot \underline{x}) + b (\underline{w} \cdot \underline{x}). && \text{(Theorem 1 in Vectors, twice)}
 \end{aligned}$$

Two special cases are used often. With

$$\begin{aligned}
 a &= b \\
 &= 1,
 \end{aligned}$$

and $1 \underline{u} = \underline{u}$, the theorem gives $\underline{x} \cdot (\underline{u} + \underline{w}) = \underline{x} \cdot \underline{u} + \underline{x} \cdot \underline{w}$. With $\underline{w} = \underline{u}$ and $b = 0$, and $0 \underline{u} = \underline{0}$, it gives $\underline{x} \cdot (a \underline{u}) = a (\underline{x} \cdot \underline{u})$, and likewise in the first slot. $\square$

Example 0.11 (Splitting a dot product). Let $\underline{x} = (1, 2)$, $\underline{u} = (3, 0)$, $\underline{w} = (0, 1)$, $a = 2$ and $b = -1$. Directly, $a \underline{u} + b \underline{w} = (6, -1)$ and

$$\begin{aligned}
 \underline{x} \cdot (6, -1) &= 6 - 2 \\
 &= 4.
 \end{aligned}$$

By Theorem 0.5,

$$\begin{aligned} a(\tilde{x} \cdot \tilde{u}) + b(\tilde{x} \cdot \tilde{w}) &= 2 \cdot 3 + (-1) \cdot 2 \\ &= 4, \end{aligned}$$

the same number.

Definition 0.5 (Orthogonal complement). The **orthogonal complement** [5] of a set $\mathcal{X}$ of vectors in $\mathbb{R}^p$ is the set of vectors orthogonal (Definition 19 in Vectors) to every vector of $\mathcal{X}$:

$$\mathcal{X}^\perp \stackrel{\text{def}}{=} \{\tilde{u} \in \mathbb{R}^p : \tilde{x} \cdot \tilde{u} = 0 \text{ for all } \tilde{x} \in \mathcal{X}\}.$$

$\mathcal{X}^\perp$ is read “ $\mathcal{X}$ perp”.

Example 0.12 (Orthogonal complements in $\mathbb{R}^3$).

- **A plane.** Let $\mathcal{X} = \{(a, b, 0) : a, b \in \mathbb{R}\}$, the plane $z = 0$. A vector $\tilde{u}$ is in $\mathcal{X}^\perp$ when

$$\begin{aligned} (a, b, 0) \cdot \tilde{u} &= au_1 + bu_2 \\ &= 0 \end{aligned}$$

for all a and b . Taking $(a, b) = (1, 0)$ forces $u_1 = 0$, and taking $(a, b) = (0, 1)$ forces $u_2 = 0$; conversely, if

$$\begin{aligned} u_1 &= u_2 \\ &= 0, \end{aligned}$$

then $au_1 + bu_2 = 0$ for all a and b . So $\mathcal{X}^\perp = \{(0, 0, c) : c \in \mathbb{R}\}$, the z -axis.

- **Two vectors.** The two-element set $\{(1, 0, 0), (0, 1, 0)\}$ is not a subspace, and its orthogonal complement is the z -axis too: the two conditions $u_1 = 0$ and $u_2 = 0$ are the same as for the plane.
- **The extremes.** $\{\tilde{0}\}^\perp = \mathbb{R}^3$, because $\tilde{0} \cdot \tilde{u} = 0$ for every $\tilde{u}$. $(\mathbb{R}^3)^\perp = \{\tilde{0}\}$, because a $\tilde{u}$ in it is orthogonal to itself, and

$$\begin{aligned} \tilde{u} \cdot \tilde{u} &= u_1^2 + u_2^2 + u_3^2 \\ &= 0 \end{aligned}$$

forces $\tilde{u} = \tilde{0}$.

i Remark (The orthogonal complement of a set that is not a subspace)

Remark 0.1 (The orthogonal complement of a set that is not a subspace). $\mathcal{X}$ in Definition 0.5 need not be a subspace.

Theorem 0.6 (An orthogonal complement is a subspace). *For any set $\mathcal{X}$ of vectors in $\mathbb{R}^p$, $\mathcal{X}^\perp$ (Definition 0.5) is a subspace of $\mathbb{R}^p$ (Definition 4 in Subspaces and Rank).*

i Proof

Proof. $\mathcal{X}^\perp$ is not empty:

$$\begin{aligned} \tilde{x} \cdot \tilde{0} &= \sum_i x_i \cdot 0 \\ &= 0 \end{aligned}$$

for every $\tilde{x}$, so $\tilde{0} \in \mathcal{X}^\perp$. Let $\tilde{u}, \tilde{w} \in \mathcal{X}^\perp$, let $c \in \mathbb{R}$, and let $\tilde{x} \in \mathcal{X}$.

Addition.

$$\begin{aligned} \tilde{x} \cdot (\tilde{u} + \tilde{w}) &= \tilde{x} \cdot \tilde{u} + \tilde{x} \cdot \tilde{w} && \text{(Theorem 0.5, special case } a \text{ and } b \text{ both equal to 1)} \\ &= 0 + 0 && (\tilde{u}, \tilde{w} \in \mathcal{X}^\perp) \\ &= 0. && \text{(arithmetic)} \end{aligned}$$

Scalar multiplication.

$$\begin{aligned} \tilde{x} \cdot (c \tilde{u}) &= c (\tilde{x} \cdot \tilde{u}) && \text{(Theorem 0.5, special case } b = 0) \\ &= c \cdot 0 && (\tilde{u} \in \mathcal{X}^\perp) \\ &= 0. && \text{(arithmetic)} \end{aligned}$$

These two equations hold for every $\tilde{x} \in \mathcal{X}$, so $\tilde{u} + \tilde{w}$ and $c \tilde{u}$ are in $\mathcal{X}^\perp$. □

Example 0.13 (The complement of a single vector is a line). The one-element set $\mathcal{X} = \{(1, 2)\}$ in $\mathbb{R}^2$ is not a subspace: $2(1, 2) = (2, 4)$ is not in it. Its orthogonal complement is

$$\begin{aligned} \mathcal{X}^\perp &= \{\tilde{u} \in \mathbb{R}^2 : u_1 + 2u_2 = 0\} \\ &= \{c(-2, 1) : c \in \mathbb{R}\}, \end{aligned}$$

since $u_1 + 2u_2 = 0$ means

$$\begin{aligned}\tilde{u} &= (-2u_2, u_2) \\ &= u_2(-2, 1).\end{aligned}$$

That set is $\text{span}\{(-2, 1)\}$, a subspace by [Theorem 2 in Subspaces and Rank](#), as [Theorem 0.6](#) says it must be. It contains $(-2, 1)$, the vector [Remark 8 in Vectors](#) found perpendicular to $(1, 2)$.

Theorem 0.7 (The orthogonal complement of a column space is a null space). *For any $m \times n$ matrix $\mathbf{A}$,*

$$\mathcal{C}(\mathbf{A})^\perp = \mathcal{N}(\mathbf{A}^\top).$$

i Proof

Proof. Let $\tilde{a}_1, \dots, \tilde{a}_n$ be the columns of $\mathbf{A}$, so $\tilde{a}_j = (a_{1j}, \dots, a_{mj})$. For $\tilde{x} \in \mathbb{R}^m$, entry j of $\mathbf{A}^\top \tilde{x}$ is

$$\begin{aligned}(\mathbf{A}^\top \tilde{x})_j &= \sum_{i=1}^m (\mathbf{A}^\top)_{ji} x_i \quad (\text{Definition 13 in Matrices}) \\ &= \sum_{i=1}^m a_{ij} x_i \quad (\text{Definition 3 in Matrices}) \\ &= \tilde{a}_j \cdot \tilde{x}. \quad (\text{Definition 8 in Vectors})\end{aligned}$$

So $\tilde{x} \in \mathcal{N}(\mathbf{A}^\top)$ ([Definition 3 in Column Space, Null Space and Rank-Nullity](#)) exactly when $\tilde{a}_j \cdot \tilde{x} = 0$ for every column $\tilde{a}_j$.

$\mathcal{C}(\mathbf{A})^\perp \subseteq \mathcal{N}(\mathbf{A}^\top)$. Each column $\tilde{a}_j$ is in $\mathcal{C}(\mathbf{A})$ ([Theorem 1 in Column Space, Null Space and Rank-Nullity](#)), so a vector orthogonal to all of $\mathcal{C}(\mathbf{A})$ is orthogonal to every column.

$\mathcal{N}(\mathbf{A}^\top) \subseteq \mathcal{C}(\mathbf{A})^\perp$. Suppose $\tilde{a}_j \cdot \tilde{x} = 0$ for every j , and take any $\tilde{y} \in \mathcal{C}(\mathbf{A})$, so $\tilde{y} = \mathbf{A}\tilde{z}$ for some $\tilde{z} \in \mathbb{R}^n$ ([Definition 1 in Column Space, Null Space and Rank-Nullity](#)). Then

$$\begin{aligned}
\underset{\sim}{y} \cdot \underset{\sim}{x} &= \sum_{i=1}^m (\mathbf{A}\underset{\sim}{z})_i x_i && \text{(Definition 8 in Vectors)} \\
&= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} z_j \right) x_i && \text{(Definition 13 in Matrices)} \\
&= \sum_{i=1}^m \sum_{j=1}^n a_{ij} z_j x_i && \text{(distribute each } x_i \text{ over the inner sum)} \\
&= \sum_{j=1}^n \sum_{i=1}^m a_{ij} z_j x_i && \text{(swap the order of the finite sums)} \\
&= \sum_{j=1}^n z_j \sum_{i=1}^m a_{ij} x_i && \text{(commute, then factor } z_j \text{ out of the inner sum)} \\
&= \sum_{j=1}^n z_j (\underset{\sim}{a}_j \cdot \underset{\sim}{x}) && \text{(Definition 8 in Vectors)} \\
&= \sum_{j=1}^n z_j \cdot 0 && \text{(each } \underset{\sim}{a}_j \cdot \underset{\sim}{x} = 0) \\
&= 0. && \text{(arithmetic)}
\end{aligned}$$

So $\underset{\sim}{x}$ is orthogonal to every vector of $\mathcal{C}(\mathbf{A})$. □

Example 0.14 (The complement of the column space in [Example 1 in Column Space, Null Space and Rank-Nullity](#)). For $\mathbf{A}$ in [Example 1 in Column Space, Null Space and Rank-Nullity](#), $\mathcal{C}(\mathbf{A}) = \text{span}\{(1, 3)\}$ (1). [Example 3 in Column Space, Null Space and Rank-Nullity](#) found $\mathcal{N}(\mathbf{A}^\top) = \text{span}\{(-3, 1)\}$, so by [Theorem 0.7](#) this line is $\mathcal{C}(\mathbf{A})^\perp$. Directly, for any c ,

$$\begin{aligned}
(c(1, 3)) \cdot (-3, 1) &= c((1, 3) \cdot (-3, 1)) && \text{(Theorem 0.5, special case } b = 0, \text{ first slot)} \\
&= c(-3 + 3) && \text{(Definition 8 in Vectors)} \\
&= 0. && \text{(arithmetic)}
\end{aligned}$$

Theorem 0.8 (A subspace and its orthogonal complement make up the whole space).
Let $\mathcal{S}$ be a subspace of $\mathbb{R}^p$. Then

1. $\mathcal{S} \cap \mathcal{S}^\perp = \{0\}$;
2. $\dim(\mathcal{S}) + \dim(\mathcal{S}^\perp) = p$;
3. $\mathbb{R}^p = \mathcal{S} \oplus \mathcal{S}^\perp$ ([Definition 0.3](#)).

So every $y \in \mathbb{R}^p$ can be written as $y = \underset{\sim}{u} + \underset{\sim}{v}$ with $\underset{\sim}{u} \in \mathcal{S}$ and $\underset{\sim}{v} \in \mathcal{S}^\perp$ in only one way ([Theorem 0.4](#)).

i Proof

Proof. $\mathcal{S}^\perp$ is a subspace (Theorem 0.6), so it has a dimension (Definition 9 in Subspaces and Rank).

Part 1. Both $\mathcal{S}$ and $\mathcal{S}^\perp$ contain $\underline{0}$ (Theorem 1 in Subspaces and Rank). If $\underline{x}$ is in both, then $\underline{x}$ is orthogonal to every vector of $\mathcal{S}$, $\underline{x}$ itself included, so

$$\begin{aligned}\underline{x} \cdot \underline{x} &= x_1^2 + \cdots + x_p^2 \\ &= 0,\end{aligned}$$

which forces every $x_i = 0$.

Part 2. Let $d = \dim(\mathcal{S})$. If $d = 0$, then $\mathcal{S}$ has a basis (Theorem 9 in Subspaces and Rank) with 0 vectors (Definition 9 in Subspaces and Rank), the empty list, whose span is $\{\underline{0}\}$ (Definition 5 in Subspaces and Rank); so $\mathcal{S} = \{\underline{0}\}$, $\mathcal{S}^\perp = \mathbb{R}^p$ (every $\underline{u}$ satisfies $\underline{0} \cdot \underline{u} = 0$), and $0 + p = p$ (Example 16 in Subspaces and Rank). If $d \geq 1$, choose a basis $\underline{a}_1, \dots, \underline{a}_d$ of $\mathcal{S}$ (Theorem 9 in Subspaces and Rank), and let $\mathbf{A}$ be the $p \times d$ matrix with these columns. The basis spans $\mathcal{S}$, so $\mathcal{S} = \mathcal{C}(\mathbf{A})$ (Theorem 1 in Column Space, Null Space and Rank-Nullity), and $\mathcal{S}^\perp = \mathcal{N}(\mathbf{A}^\top)$ (Theorem 0.7). $\mathbf{A}^\top$ is $d \times p$, so

$$\begin{aligned}\dim(\mathcal{S}^\perp) &= \dim(\mathcal{N}(\mathbf{A}^\top)) && \text{(Theorem 0.7)} \\ &= \text{nullity}(\mathbf{A}^\top) && \text{(Definition 7 in Column Space, Null Space and Rank-Nullity)} \\ &= p - \text{rank}(\mathbf{A}^\top) && \text{(Theorem 7 in Column Space, Null Space and Rank-Nullity, for the } p \text{ columns of } \mathbf{A}^\top) \\ &= p - \text{rank}(\mathbf{A}) && \text{(Theorem 9 in Column Space, Null Space and Rank-Nullity)} \\ &= p - \dim(\mathcal{C}(\mathbf{A})) && \text{(Theorem 6 in Column Space, Null Space and Rank-Nullity)} \\ &= p - d. && (\mathcal{C}(\mathbf{A}) = \mathcal{S})\end{aligned}$$

Part 3. $\mathcal{S} + \mathcal{S}^\perp$ is a subspace of $\mathbb{R}^p$ (Theorem 0.1), and

$$\begin{aligned}\dim(\mathcal{S} + \mathcal{S}^\perp) &= \dim(\mathcal{S}) + \dim(\mathcal{S}^\perp) - \dim(\mathcal{S} \cap \mathcal{S}^\perp) && \text{(Theorem 0.2)} \\ &= p - \dim(\mathcal{S} \cap \mathcal{S}^\perp) && \text{(part 2)} \\ &= p - 0 && \text{(part 1, and } \dim(\{\underline{0}\}) = 0 \text{ by Example 16 in Subspaces and Rank)} \\ &= p. && \text{(arithmetic)}\end{aligned}$$

Since $\mathcal{S} + \mathcal{S}^\perp \subseteq \mathbb{R}^p$, $\mathbb{R}^p$ is a subspace (Example 3 in Subspaces and Rank) and $\dim(\mathbb{R}^p) = p$ (Example 16 in Subspaces and Rank), Theorem 11 in Subspaces and Rank gives $\mathcal{S} + \mathcal{S}^\perp = \mathbb{R}^p$. Together with part 1, $\mathbb{R}^p = \mathcal{S} \oplus \mathcal{S}^\perp$. The uniqueness of the split then follows from Theorem 0.4 (statement 1 implies statement 3). $\square$

Example 0.15 (Splitting a vector of $\mathbb{R}^3$ along a line and its complement). Let $\mathcal{S} = \text{span}\{(1, 1, 0)\}$ in $\mathbb{R}^3$. A single nonzero vector is linearly independent ($c \tilde{v} = \tilde{0}$ with $\tilde{v} \neq \tilde{0}$ forces $c = 0$), so $(1, 1, 0)$ is a basis of $\mathcal{S}$ and $\dim(\mathcal{S}) = 1$. A vector $\tilde{u}$ is in $\mathcal{S}^\perp$ exactly when $u_1 + u_2 = 0$: that condition says $(1, 1, 0) \cdot \tilde{u} = 0$, and then

$$\begin{aligned} c(1, 1, 0) \cdot \tilde{u} &= c(u_1 + u_2) \\ &= 0 \end{aligned}$$

for every c . So

$$\begin{aligned} \mathcal{S}^\perp &= \{(a, -a, c) : a, c \in \mathbb{R}\} \\ &= \text{span}\{(1, -1, 0), (0, 0, 1)\}. \end{aligned}$$

Those two vectors are linearly independent, because $a(1, -1, 0) + c(0, 0, 1) = (a, -a, c)$ is $\tilde{0}$ only if

$$\begin{aligned} a &= c \\ &= 0, \end{aligned}$$

so $\dim(\mathcal{S}^\perp) = 2$, and $1 + 2 = 3$, as part 2 says.

To split $\tilde{y} = (3, 1, 2)$, look for $\tilde{u} = t(1, 1, 0)$ with

$$\begin{aligned} \tilde{v} &= \tilde{y} - \tilde{u} \\ &= (3 - t, 1 - t, 2) \end{aligned}$$

in $\mathcal{S}^\perp$: that membership needs $(3 - t) + (1 - t) = 0$, so $t = 2$. Then $\tilde{u} = (2, 2, 0)$ and $\tilde{v} = (1, -1, 2)$, with $\tilde{u} + \tilde{v} = (3, 1, 2)$ and

$$\begin{aligned} (1, 1, 0) \cdot (1, -1, 2) &= 1 - 1 + 0 \\ &= 0. \end{aligned}$$

Theorem 0.9 (The orthogonal complement of the orthogonal complement). *For any subspace $\mathcal{S}$ of $\mathbb{R}^p$,*

$$(\mathcal{S}^\perp)^\perp = \mathcal{S}.$$

i Proof

Proof. $\mathcal{S} \subseteq (\mathcal{S}^\perp)^\perp$. Take $\tilde{s} \in \mathcal{S}$ and any $\tilde{v} \in \mathcal{S}^\perp$. By Definition 0.5, $\tilde{s} \cdot \tilde{v} = 0$, so $\tilde{v} \cdot \tilde{s} = 0$ too (Theorem 1 in Vectors). So $\tilde{s}$ is orthogonal to every vector of $\mathcal{S}^\perp$, that is, $\tilde{s} \in (\mathcal{S}^\perp)^\perp$.

Equal dimensions. $\mathcal{S}^\perp$ is a subspace (Theorem 0.6), so part 2 of Theorem 0.8 applies to it as well as to $\mathcal{S}$:

$$\begin{aligned}
\dim((\mathcal{S}^\perp)^\perp) &= p - \dim(\mathcal{S}^\perp) && \text{(Theorem 0.8, applied to } \mathcal{S}^\perp) \\
&= p - (p - \dim(\mathcal{S})) && \text{(Theorem 0.8, applied to } \mathcal{S}) \\
&= \dim(\mathcal{S}). && \text{(arithmetic)}
\end{aligned}$$

$(\mathcal{S}^\perp)^\perp$ is a subspace (Theorem 0.6) that contains $\mathcal{S}$ and has the same dimension, so it equals $\mathcal{S}$ (Theorem 11 in Subspaces and Rank). $\square$

Example 0.16 (Back to the line). In Example 0.15, $\mathcal{S} = \text{span}\{(1, 1, 0)\}$ and $\mathcal{S}^\perp = \text{span}\{(1, -1, 0), (0, 0, 1)\}$. Put those two spanning vectors in the columns of the 3×2 matrix $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\mathcal{S}^\perp = \mathcal{C}(\mathbf{B})$ (Theorem 1 in Column Space, Null Space and Rank-Nullity) and $(\mathcal{S}^\perp)^\perp = \mathcal{N}(\mathbf{B}^\top)$ (Theorem 0.7). Since $\mathbf{B}^\top \tilde{w} = (w_1 - w_2, w_3)$ (Definition 13 in Matrices), $\tilde{w} \in (\mathcal{S}^\perp)^\perp$ exactly when $w_1 - w_2 = 0$ and $w_3 = 0$. So

$$\begin{aligned}
(\mathcal{S}^\perp)^\perp &= \{(a, a, 0) : a \in \mathbb{R}\} \\
&= \mathcal{S}.
\end{aligned}$$

Example 0.17 (A set that is not a subspace does not come back). The one-element set $\mathcal{X} = \{(1, 1, 0)\}$ is not a subspace. Its orthogonal complement is the same as that of $\mathcal{S}$ in Example 0.16, since both are defined by the single condition $u_1 + u_2 = 0$, so $(\mathcal{X}^\perp)^\perp = \text{span}\{(1, 1, 0)\}$, which contains $(2, 2, 0) \notin \mathcal{X}$. Theorem 0.9 needs $\mathcal{S}$ to be a subspace.

The fundamental theorem of linear algebra

Theorem 0.10 (Fundamental theorem of linear algebra). *Let $\mathbf{A}$ be an $m \times n$ matrix with $\text{rank}(\mathbf{A}) = r$. Then*

1. $\mathcal{C}(\mathbf{A})^\perp = \mathcal{N}(\mathbf{A}^\top)$, and $\mathbb{R}^m = \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\top)$;

2. $\mathcal{C}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^\top)^\perp$;

3. $\mathcal{N}(\mathbf{A})^\perp = \mathcal{C}(\mathbf{A}^\top)$, and $\mathbb{R}^n = \mathcal{C}(\mathbf{A}^\top) \oplus \mathcal{N}(\mathbf{A})$;

- 4.

$$\begin{aligned}\dim(\mathcal{C}(\mathbf{A})) &= \dim(\mathcal{C}(\mathbf{A}^\top)) \\ &= r,\end{aligned}$$

$$\dim(\mathcal{N}(\mathbf{A})) = n - r, \text{ and } \dim(\mathcal{N}(\mathbf{A}^\top)) = m - r.$$

i Proof

Proof. Part 1. The first equation is Theorem 0.7. $\mathcal{C}(\mathbf{A})$ is a subspace of $\mathbb{R}^m$ (Theorem 1 in Column Space, Null Space and Rank-Nullity), so

$$\begin{aligned}\mathbb{R}^m &= \mathcal{C}(\mathbf{A}) \oplus \mathcal{C}(\mathbf{A})^\perp \quad (\text{Theorem 0.8}) \\ &= \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\top). \quad (\text{first equation})\end{aligned}$$

Part 2.

$$\begin{aligned}\mathcal{C}(\mathbf{A}) &= (\mathcal{C}(\mathbf{A})^\perp)^\perp \quad (\text{Theorem 0.9}) \\ &= \mathcal{N}(\mathbf{A}^\top)^\perp. \quad (\text{part 1})\end{aligned}$$

Part 3. Apply parts 2 and 1 to the $n \times m$ matrix $\mathbf{A}^\top$, using $(\mathbf{A}^\top)^\top = \mathbf{A}$: part 2 gives $\mathcal{C}(\mathbf{A}^\top) = \mathcal{N}(\mathbf{A})^\perp$, and part 1 gives $\mathbb{R}^n = \mathcal{C}(\mathbf{A}^\top) \oplus \mathcal{N}(\mathbf{A})$.

Part 4. $\dim(\mathcal{C}(\mathbf{A})) = r$ by Theorem 6 in Column Space, Null Space and Rank-Nullity. $\dim(\mathcal{C}(\mathbf{A}^\top)) = \text{rank}(\mathbf{A}^\top)$ by Theorem 6 in Column Space, Null Space and Rank-Nullity, and that is r by Theorem 9 in Column Space, Null Space and Rank-Nullity. $\mathbf{A}$ has n columns, so $\dim(\mathcal{N}(\mathbf{A})) = n - r$ (Definition 7 in Column Space, Null Space and Rank-Nullity, Theorem 7 in Column Space, Null Space and Rank-Nullity). $\mathbf{A}^\top$ has m columns and rank r , so $\dim(\mathcal{N}(\mathbf{A}^\top)) = m - r$ in the same way. $\square$

Example 0.18 (The four subspaces of the matrix in Example 1 in Column Space, Null Space and Rank-Nullity). For $\mathbf{A}$ in Example 1 in Column Space, Null Space and Rank-Nullity, $m = 2$, $n = 3$ and $r = 1$ (Example 10 in Column Space, Null Space and Rank-Nullity).

- **In $\mathbb{R}^3$:** the row space $\mathcal{C}(\mathbf{A}^\top) = \text{span}\{(1, -2, -2)\}$

(1) has dimension $1 = r$, and $\mathcal{N}(\mathbf{A}) = \text{span}\{(2, 1, 0), (2, 0, 1)\}$ has dimension $2 = n - r$ (Example 11 in Column Space, Null Space and Rank-Nullity). The spanning vectors are orthogonal:

$$\begin{aligned}(1, -2, -2) \cdot (2, 1, 0) &= 2 - 2 + 0 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(1, -2, -2) \cdot (2, 0, 1) &= 2 + 0 - 2 \\ &= 0.\end{aligned}$$

- **In $\mathbb{R}^2$:** $\mathcal{C}(\mathbf{A}) = \text{span}\{(1, 3)\}$ has dimension $1 = r$, and $\mathcal{N}(\mathbf{A}^\top) = \text{span}\{(-3, 1)\}$ has dimension $1 = m - r$ (Example 0.14).

By part 3, $y = (1, 0, 0)$ splits uniquely into a row-space part $t(1, -2, -2)$ and a null-space part. The null-space part $y - t(1, -2, -2)$ lies in $\mathcal{N}(\mathbf{A})$ (part 3), and $\mathcal{N}(\mathbf{A}) = \mathcal{C}(\mathbf{A}^\top)^\perp$ (Theorem 0.7 applied to $\mathbf{A}^\top$, with $(\mathbf{A}^\top)^\top = \mathbf{A}$), so it must be orthogonal to $(1, -2, -2)$:

$$\begin{aligned}0 &= (1, -2, -2) \cdot ((1, 0, 0) - t(1, -2, -2)) && \text{(orthogonality)} \\ &= (1, -2, -2) \cdot (1 \cdot (1, 0, 0) + (-t)(1, -2, -2)) && \text{(write the difference as a linear combination)} \\ &= 1 \cdot ((1, -2, -2) \cdot (1, 0, 0)) + (-t)((1, -2, -2) \cdot (1, -2, -2)) && \text{(Theorem 0.5)} \\ &= 1 \cdot 1 + (-t) \cdot 9 && \text{(Definition 8 in Vectors)} \\ &= 1 - 9t, && \text{(arithmetic)}\end{aligned}$$

so $t = \frac{1}{9}$. The row-space part is $(\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and the null-space part is $(\frac{8}{9}, \frac{2}{9}, \frac{2}{9})$; as a check, row 1 of $\mathbf{A}$ gives $\frac{8}{9} - 2 \cdot \frac{2}{9} - 2 \cdot \frac{2}{9} = 0$, and row 2 is 3 times row 1, so $\mathbf{A}$ sends the null-space part to $\mathbf{0}_2$.

These four statements are the fundamental theorem of linear algebra [6].

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