Determinants

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Definition

Definition 0.1 (Determinant)**.** The **determinant** [1] of a $p \times p$ matrix $\mathbf{A}$ with entries a_{ij}, written $\det(\mathbf{A})$ or $|\mathbf{A}|$, is the number defined recursively in p:

- For $p = 1$, $\det(\mathbf{A}) = a_{11}$.
- For $p \geq 2$,
$$\det(\mathbf{A}) = \sum_{j=1}^{p} (-1)^{1+j}\, a_{1j} \det\big(\mathbf{A}_{(1j)}\big),$$
where $\mathbf{A}_{(1j)}$ is the $(p-1) \times (p-1)$ matrix left after deleting row 1 and column j of $\mathbf{A}$.

Example 0.1 (Determinant of a 2×2 matrix)**.** For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, deleting row 1 and column 1 leaves $\big[d\big]$, and deleting row 1 and column 2 leaves $\big[c\big]$. So:

$$\begin{aligned} \det(\mathbf{A}) &= (-1)^{1+1}\, a \det\big(\big[d\big]\big) + (-1)^{1+2}\, b \det\big(\big[c\big]\big) && \text{(definition, } p = 2\text{)} \\ &= ad - bc && \text{(definition, } p = 1\text{)} \end{aligned}$$

For example,

$$\begin{aligned} \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) &= 2 \cdot 2 - 1 \cdot 1 \\ &= 3. \end{aligned}$$

Remark (Cofactor expansion)

Remark 0.1 (Cofactor expansion). The recursive formula in Definition 0.1 is the *cofactor expansion* along the first row. Other sources define the determinant as a sum over all orderings of the columns and derive this expansion from it; the two definitions agree [2]. Each ordering picks one entry from each row, from the column that the ordering assigns to that row, and multiplies them. Its sign is $+1$ if the ordering takes an even number of swaps of two columns to reach from $(1, 2, \dots, p)$, and -1 if it takes an odd number.
For $p = 2$, the columns have two orderings. The ordering $(1, 2)$ keeps each column in place, picks the entries a_{11} and a_{22}, and has sign $+1$. The ordering $(2, 1)$ swaps the two columns, picks the entries a_{12} and a_{21}, and has sign -1. The sum $a_{11}a_{22} - a_{12}a_{21}$ is the value $ad - bc$ from Example 0.1.

See also [1].

Video

Chapter 6 of the *Essence of linear algebra* series by 3Blue1Brown shows the geometric meaning of the determinant:

- The determinant shows how the determinant measures the factor by which a linear transformation scales areas and volumes, and why a zero determinant corresponds to compressing space into a lower dimension.

Diagonal matrices

Theorem 0.1 (Determinant of a diagonal matrix)**.** *The determinant (Definition 0.1) of a $p \times p$ diagonal matrix (Definition 4 in Special Matrices, Quadratic Forms and Trace) is the product of its diagonal entries:*

$$\det\left(\begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_p \end{bmatrix}\right) = \prod_{i=1}^{p} d_i$$

Proof

Proof. By induction on p. For $p = 1$, the matrix is $\begin{bmatrix} d_1 \end{bmatrix}$, and its determinant is d_1 by definition.
For $p \geq 2$, suppose the result holds for $(p-1) \times (p-1)$ diagonal matrices, and let $\mathbf{D}$ be

$p \times p$ and diagonal. Row 1 of $\mathbf{D}$ is $(d_1, 0, \ldots, 0)$, so only the $j = 1$ term of the definition can be nonzero, and deleting row 1 and column 1 of $\mathbf{D}$ leaves the diagonal matrix with diagonal entries $d_2, \ldots, d_p$:

$$\begin{aligned}
\det(\mathbf{D}) &= (-1)^{1+1}\, d_1 \det\left(\mathbf{D}_{(11)}\right) + \sum_{j=2}^{p} (-1)^{1+j} \cdot 0 \cdot \det\left(\mathbf{D}_{(1j)}\right) && \text{(definition)} \\
&= d_1 \det\left(\mathbf{D}_{(11)}\right) && \text{(drop the zero terms)} \\
&= d_1 \prod_{i=2}^{p} d_i && \text{(induction hypothesis)} \\
&= \prod_{i=1}^{p} d_i && \text{(combine)}
\end{aligned}$$

□

Example 0.1 (Determinant of a scaled identity matrix)**.** For a number c, $c\,\mathbf{I}_p$ is diagonal with every diagonal entry equal to c, so $\det(c\,\mathbf{I}_p) = c^p$. In particular, $\det(\mathbf{I}_p) = 1$.

Matrix products

Theorem 0.1 (Determinant of a product)**.** *For $p \times p$ matrices $\mathbf{A}$ and $\mathbf{B}$,*

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B}).$$

Example 0.1 (Checking the product rule)**.** For $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, Example 0.1 gives

$$\begin{aligned}
\det(\mathbf{A}) &= 2 \cdot 1 - 1 \cdot 0 \\
&= 2
\end{aligned}$$

and

$$\begin{aligned}
\det(\mathbf{B}) &= 1 \cdot 1 - 0 \cdot 3 \\
&= 1.
\end{aligned}$$

The product is

$$\mathbf{AB} = \begin{bmatrix} 2 \cdot 1 + 1 \cdot 3 & 2 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 3 & 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix},$$

with

$$\begin{aligned} \det(\mathbf{AB}) &= 5 \cdot 1 - 1 \cdot 3 \\ &= 2 \\ &= \det(\mathbf{A}) \det(\mathbf{B}). \end{aligned}$$

Remark (An example is not a proof)

Remark 0.1 (An example is not a proof). Example 0.1 checks Theorem 0.1 for one pair of 2×2 matrices, which shows the theorem holds there but does not prove it for every pair. The proof needs properties of the determinant that these notes don't develop; see Banerjee and Roy [2].

See also [1].

Eigenvalues

Theorem 0.1 (The determinant of a symmetric matrix is the product of its eigenvalues)**.** *Let* $\mathbf{A}$ *be a* $p \times p$ *symmetric matrix with real entries, with eigendecomposition* $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\top$ *(Definition 3 in Eigenvalues and Matrix Decompositions) and eigenvalues* $\lambda_1, \ldots, \lambda_p$ *on the diagonal of* $\mathbf{\Lambda}$*. Then*

$$\det(\mathbf{A}) = \prod_{i=1}^{p} \lambda_i.$$

Proof

Proof. First, $\det(\mathbf{Q}^\top) \det(\mathbf{Q}) = 1$:

$$\begin{aligned} \det(\mathbf{Q}^\top) \det(\mathbf{Q}) &= \det(\mathbf{Q}^\top \mathbf{Q}) && \text{(determinant of a product)} \\ &= \det(\mathbf{I}_p) && (\mathbf{Q} \text{ is orthogonal}) \\ &= 1 && \text{(determinant of the identity)} \end{aligned}$$

Then:

$$
\begin{aligned}
\det(\mathbf{A}) &= \det(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\top) && \text{(substitute the eigendecomposition)} \\
&= \det(\mathbf{Q})\det(\mathbf{\Lambda})\det(\mathbf{Q}^\top) && \text{(determinant of a product, twice)} \\
&= \det(\mathbf{\Lambda}) \cdot \det(\mathbf{Q}^\top)\det(\mathbf{Q}) && \text{(reorder the three numbers)} \\
&= \det(\mathbf{\Lambda}) && \text{(the first display)} \\
&= \prod_{i=1}^{p} \lambda_i && \text{(determinant of a diagonal matrix)}
\end{aligned}
$$

The product steps are Theorem 0.1, the orthogonality step is Definition 10 in Special Matrices, Quadratic Forms and Trace, and the identity and diagonal steps are Theorem 0.1 and Example 0.1. □

Corollary 0.1 (A positive definite matrix has a positive determinant)**.** *If* $\mathbf{A}$ *is positive definite (Definition 2 in Definite Matrices), then* $\det(\mathbf{A}) > 0$.

Proof

Proof. By Theorem 2 in Definite Matrices, every eigenvalue of $\mathbf{A}$ is positive, so their product, which is $\det(\mathbf{A})$ by Theorem 0.1, is positive. □

Example 0.1 (Two ways to the same determinant)**.** $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues 3 and 1 (Example 1 in Eigenvalues and Matrix Decompositions), so Theorem 0.1 gives

$$
\begin{aligned}
\det(\mathbf{A}) &= 3 \cdot 1 \\
&= 3,
\end{aligned}
$$

matching $2 \cdot 2 - 1 \cdot 1 = 3$ from Example 0.1.

[1] Wikipedia contributors, "Determinant — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Determinant

[2] S. Banerjee and A. Roy, *Linear algebra and matrix analysis for statistics*, vol. 181. Crc Press Boca Raton, 2014. Available: https://www.routledge.com/Linear-Algebra-and-Matrix-Analysis-for-Statistics/Banerjee-Roy/p/book/9781420095388