

Complex Vectors and Matrices

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Complex vectors and matrices

Definition 0.1 (Complex vector). A **complex vector** of length p is a column vector whose entries are [complex numbers](#); the set of all of them is denoted $\mathbb{C}^p$.

Definition 0.2 (Complex matrix). A **complex matrix** of size $m \times n$ is a matrix with complex entries; the set of all of them is denoted $\mathbb{C}^{m \times n}$.

Example 0.1 (A complex vector and its conjugate).

$$\underset{3 \times 1}{\mathbb{z}} = \begin{bmatrix} 1+i \\ 2 \\ -3i \end{bmatrix} \in \mathbb{C}^3, \quad \underset{3 \times 1}{\bar{\mathbb{z}}} = \begin{bmatrix} 1-i \\ 2 \\ 3i \end{bmatrix}.$$

The real entry 2 is its own conjugate. Every real vector is also a complex vector, since each real number a is the complex number $a + 0i$, so $\mathbb{R}^p$ is a subset of $\mathbb{C}^p$.

Definition 0.3 (Complex conjugate of a complex vector or matrix). The **complex conjugate** of a complex vector or matrix takes the [complex conjugate](#) of each entry: $(\bar{\mathbf{A}})_{jk} \stackrel{\text{def}}{=} \overline{a_{jk}}$.

Definition 0.4 (Conjugate transpose, Hermitian adjoint, Hermitian conjugate, transjugate). The **conjugate transpose** [1] (also called the **Hermitian adjoint**, **Hermitian conjugate**, or **transjugate**) of an $m \times n$ complex matrix $\mathbf{A}$ is the $n \times m$ matrix $\mathbf{A}^H$ obtained by transposing $\mathbf{A}$ and then taking the complex conjugate of each entry:

$$(\mathbf{A}^H)_{jk} \stackrel{\text{def}}{=} \overline{a_{kj}} \quad \text{for } j \in \{1, \dots, n\}, \quad k \in \{1, \dots, m\}.$$

i Remark (Notation for the conjugate transpose)

Remark 0.1 (Notation for the conjugate transpose). The operation is denoted variously across mathematics and physics: $\mathbf{A}^*$ in pure mathematics (Axler [2], Definition 7.7, p. 231), $\mathbf{A}^\dagger$ (“ A -dagger”) in physics and quantum mechanics, and $\mathbf{A}^H$ in engineering, numerical linear algebra, and statistics. These notes write $\mathbf{A}^H$ because $*$ is reserved for optimal points (as in $\tilde{x}^*$) and † frequently denotes the Moore-Penrose pseudoinverse in the broader literature.

The Hermitian adjoint matrix should not be confused with the classical *adjugate matrix* (the transpose of the matrix of cofactors), which older texts occasionally referred to as the “adjoint.”

Example 0.2 (The conjugate transpose of a 2×3 matrix). For

$$\underbrace{\mathbf{A}}_{2 \times 3} = \begin{bmatrix} 1+i & 2 & 0 \\ -i & 3-2i & 4 \end{bmatrix},$$

transposing gives a 3×2 matrix, and conjugating each of its entries gives

$$\underbrace{\mathbf{A}^H}_{3 \times 2} = \begin{bmatrix} 1-i & i \\ 2 & 3+2i \\ 0 & 4 \end{bmatrix}.$$

For instance,

$$\begin{aligned} (\mathbf{A}^H)_{12} &= \overline{a_{21}} \\ &= \overline{-i} \\ &= i. \end{aligned}$$

The transpose alone would have $-i$ in that position, so for a matrix with an entry that is not real, $\mathbf{A}^H$ and $\mathbf{A}^\top$ differ. For a real matrix, conjugating changes nothing, so $\mathbf{A}^H = \mathbf{A}^\top$.

i Remark (Conjugate transpose of a complex vector)

Remark 0.2 (Conjugate transpose of a complex vector). The conjugate transpose of a column vector $\tilde{z} \in \mathbb{C}^p$ is the $1 \times p$ row vector $\tilde{z}^H = [\overline{z_1}, \dots, \overline{z_p}]$.

Theorem 0.1 (Properties of the Hermitian adjoint). *Let $\mathbf{A}$ and $\mathbf{B}$ be complex matrices of compatible dimensions, and let $c \in \mathbb{C}$ be a scalar. The Hermitian adjoint (Definition 0.4) satisfies:*

1. *Involution:*

$$(\mathbf{A}^H)^H = \mathbf{A}$$

2. *Additivity:*

$$(\mathbf{A} + \mathbf{B})^H = \mathbf{A}^H + \mathbf{B}^H$$

3. *Conjugate-homogeneity:*

$$(c \mathbf{A})^H = \bar{c} \mathbf{A}^H$$

4. *Reversal of products:*

$$(\mathbf{AB})^H = \mathbf{B}^H \mathbf{A}^H$$

If $\mathbf{A}$ is invertible, then $\mathbf{A}^H$ is also invertible, and

$$(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H.$$

i Proof

Proof. Properties 1-3 follow directly from entrywise conjugation and transposition:

- For involution:

$$\begin{aligned} [(\mathbf{A}^H)^H]_{jk} &= \overline{(\mathbf{A}^H)_{kj}} \\ &= \overline{\overline{a_{jk}}} \\ &= a_{jk}. \end{aligned}$$

- For additivity:

$$\begin{aligned} [(\mathbf{A} + \mathbf{B})^H]_{jk} &= \overline{(\mathbf{A} + \mathbf{B})_{kj}} \\ &= \overline{a_{kj} + b_{kj}} \\ &= \overline{a_{kj}} + \overline{b_{kj}} \\ &= (\mathbf{A}^H)_{jk} + (\mathbf{B}^H)_{jk}. \end{aligned}$$

- For conjugate-homogeneity:

$$\begin{aligned} [(c \mathbf{A})^H]_{jk} &= \overline{(c \mathbf{A})_{kj}} \\ &= \overline{\bar{c} a_{kj}} \\ &= \bar{c} \overline{a_{kj}} \\ &= \bar{c} (\mathbf{A}^H)_{jk}. \end{aligned}$$

Unlike the real transpose where $(c \mathbf{A})^\top = c \mathbf{A}^\top$, scaling conjugates the scalar.

For property 4 (reversal of products), let $\mathbf{A}$ be an $m \times k$ matrix and $\mathbf{B}$ a $k \times n$ matrix. Entry (i, j) of the product's Hermitian adjoint (for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$) is:

$$\begin{aligned}
[(\mathbf{AB})^H]_{ij} &= \overline{(\mathbf{AB})_{ji}} && \text{(Definition 0.4)} \\
&= \overline{\sum_{s=1}^k a_{js} b_{si}} && \text{(definition of matrix multiplication)} \\
&= \sum_{s=1}^k \overline{a_{js}} \overline{b_{si}} && \text{(conjugate of sums and products)} \\
&= \sum_{s=1}^k \overline{b_{si}} \overline{a_{js}} && \text{(complex multiplication is commutative)} \\
&= \sum_{s=1}^k (\mathbf{B}^H)_{is} (\mathbf{A}^H)_{sj} && \text{(Definition 0.4)} \\
&= (\mathbf{B}^H \mathbf{A}^H)_{ij} && \text{(definition of matrix multiplication).}
\end{aligned}$$

For the inverse, applying property 4 to $\mathbf{AA}^{-1} = \mathbf{I}$ gives:

$$\begin{aligned}
(\mathbf{A}^{-1})^H \mathbf{A}^H &= (\mathbf{AA}^{-1})^H \\
&= \mathbf{I}^H \\
&= \mathbf{I},
\end{aligned}$$

so $(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H$. □

Theorem 0.2 (The adjoint property for inner products). *Let $\mathbf{A}$ be an $m \times n$ complex matrix. Under the standard inner product on complex Euclidean space, $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle \stackrel{\text{def}}{=} \underset{\sim}{u}^H \underset{\sim}{v}$,*

$$\langle \mathbf{A}\underset{\sim}{u}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{u}, \mathbf{A}^H \underset{\sim}{v} \rangle \quad \text{for all } \underset{\sim}{u} \in \mathbb{C}^n \text{ and } \underset{\sim}{v} \in \mathbb{C}^m.$$

Proof

Proof. By the definition of the standard complex inner product and the product rule for the Hermitian adjoint (Theorem 0.1):

$$\begin{aligned}
\langle \mathbf{A}\underset{\sim}{u}, \underset{\sim}{v} \rangle &= (\mathbf{A}\underset{\sim}{u})^H \underset{\sim}{v} && \text{(definition of standard inner product)} \\
&= (\underset{\sim}{u}^H \mathbf{A}^H) \underset{\sim}{v} && \text{(Theorem 0.1, reversal of products)} \\
&= \underset{\sim}{u}^H (\mathbf{A}^H \underset{\sim}{v}) && \text{(associativity of matrix multiplication)} \\
&= \langle \underset{\sim}{u}, \mathbf{A}^H \underset{\sim}{v} \rangle && \text{(definition of standard inner product).}
\end{aligned}$$
□

i Remark (Where the name “adjoint” comes from)

Remark 0.3 (Where the name “adjoint” comes from). This identity is the origin of the name **adjoint**: in functional analysis and operator theory, the *adjoint* of a linear map $T : V \rightarrow W$ between inner product spaces is defined as the unique operator $T^* : W \rightarrow V$ satisfying $\langle Tu, v \rangle = \langle u, T^*v \rangle$ (Axler [2], Definition 7.1, p. 226). When V and W are finite-dimensional spaces equipped with their standard orthonormal bases, the matrix representing the adjoint operator T^* is precisely the Hermitian adjoint matrix $\mathbf{A}^H$ (Axler [2], Proposition 7.9, p. 231).

Theorem 0.3 ($\tilde{z}^H \tilde{z}$ is a sum of squared absolute values). For every $z \in \mathbb{C}^p$,

$$\tilde{z}_{1 \times p}^H \tilde{z}_{p \times 1} = \sum_{j=1}^p |z_j|^2,$$

which is a nonnegative real number, and is 0 only when $z = \mathbf{0}_{p \times 1}$.

i Proof

Proof. The last step uses the [identity for a complex number times its conjugate](#).

$$\begin{aligned} \tilde{z}^H \tilde{z} &= \sum_{j=1}^p \overline{z_j} z_j \quad (\text{Definition 0.4, and the row-times-column product}) \\ &= \sum_{j=1}^p z_j \overline{z_j} \quad (\text{complex multiplication is commutative}) \\ &= \sum_{j=1}^p |z_j|^2 \quad (\text{a number times its conjugate}) \end{aligned}$$

By the definition of the [absolute value](#), each $|z_j|^2 = (\operatorname{Re} z_j)^2 + (\operatorname{Im} z_j)^2$ is a nonnegative real number, so their sum is too. A sum of nonnegative numbers is 0 only when every term is 0, and $|z_j|^2 = 0$ only when

$$\begin{aligned} \operatorname{Re} z_j &= \operatorname{Im} z_j \\ &= 0, \end{aligned}$$

that is, when $z_j = 0$. □

Example 0.3 (Computing $\tilde{z}^H \tilde{z}$). Let $z = (1 + i, 2)^\top$. By Theorem 0.3,

$$\begin{aligned}
\underset{1 \times 2}{\widetilde{z}}^H \underset{2 \times 1}{\widetilde{z}} &= |1+i|^2 + |2|^2 && \text{(Theorem 0.3)} \\
&= (1^2 + 1^2) + (2^2 + 0^2) && \text{(definition of the absolute value, squared)} \\
&= 2 + 4 && \text{(add inside each group)} \\
&= 6 && \text{(add)}
\end{aligned}$$

Multiplying out directly gives the same number: $\bar{2} \cdot 2 = 4$, and

$$\begin{aligned}
\overline{(1+i)}(1+i) &= (1-i)(1+i) && \text{(conjugate)} \\
&= 1+i-i-i^2 && \text{(distribute)} \\
&= 1-i^2 && \text{(cancel } i) \\
&= 2 && (i^2 = -1)
\end{aligned}$$

Example 0.4 (Why complex vectors use the conjugate transpose). For a real vector x , $x^\top x = \sum_{j=1}^p x_j^2$ is 0 only when x is the zero vector. The same formula fails for complex vectors. Let $z = (1, i)^\top$. Then

$$\begin{aligned}
\underset{1 \times 2}{\widetilde{z}}^\top \underset{2 \times 1}{\widetilde{z}} &= 1 \cdot 1 + i \cdot i && \text{(multiply entry by entry and add)} \\
&= 1 + i^2 && \text{(multiply)} \\
&= 0 && (i^2 = -1)
\end{aligned}$$

although $z \neq 0_{2 \times 1}$. Using the conjugate transpose instead,

$$\begin{aligned}
\underset{1 \times 2}{\widetilde{z}}^H \underset{2 \times 1}{\widetilde{z}} &= \bar{1} \cdot 1 + \bar{i} \cdot i && \text{(Definition 0.4)} \\
&= 1 + (-i) \cdot i && \text{(conjugate each entry)} \\
&= 1 - i^2 && \text{(multiply)} \\
&= 2 && (i^2 = -1)
\end{aligned}$$

which is positive, as Theorem 0.3 guarantees.

Definition 0.5 (Hermitian matrix, self-adjoint matrix). A square complex matrix $A \in \mathbb{C}^{n \times n}$ is **Hermitian** [3] (or **self-adjoint**) if it equals its conjugate transpose (Definition 0.4):

$$A^H = A.$$

i Remark (Why “self-adjoint”)

Remark 0.4 (Why “self-adjoint”). The term *self-adjoint* comes directly from Theorem 0.2: when $\mathbf{A}$ is Hermitian, $\langle \mathbf{A}\underline{u}, \underline{v} \rangle = \langle \underline{u}, \mathbf{A}\underline{v} \rangle$ for all $\underline{u}, \underline{v} \in \mathbb{C}^n$. The linear operator moves across the inner product without change.

Example 0.5 (A Hermitian matrix).

$$\underline{\mathbf{A}}_{2 \times 2} = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}$$

is Hermitian: transposing swaps the off-diagonal entries, and conjugating turns $1+i$ back into $1-i$ and $1-i$ back into $1+i$, so $\mathbf{A}^H = \mathbf{A}$. The diagonal entries of a Hermitian matrix must be real, because $a_{jj} = \overline{a_{jj}}$, and only a real number is its own conjugate: if $a + bi = a - bi$, then the imaginary parts give $b = -b$, so $b = 0$. Every real symmetric matrix is Hermitian, because for a real matrix $\mathbf{A}^H = \mathbf{A}^T$.

Example 0.6 (A symmetric complex matrix that is not Hermitian).

$$\underline{\mathbf{B}}_{2 \times 2} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

is symmetric, $\mathbf{B}^T = \mathbf{B}$, but it is not Hermitian:

$$\underline{\mathbf{B}}_{2 \times 2}^H = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \neq \mathbf{B}.$$

Definition 0.6 (Skew-Hermitian matrix, anti-Hermitian matrix). A square complex matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ is **skew-Hermitian** [4] (or **anti-Hermitian**) if it equals the negative of its conjugate transpose (Definition 0.4):

$$\mathbf{A}^H = -\mathbf{A}.$$

i Remark (The diagonal of a skew-Hermitian matrix)

Remark 0.5 (The diagonal of a skew-Hermitian matrix). Every entry of a skew-Hermitian matrix satisfies $a_{jk} = -\overline{a_{kj}}$. In particular, its diagonal entries satisfy $a_{jj} = -\overline{a_{jj}}$, which means

$$\begin{aligned} a_{jj} + \overline{a_{jj}} &= 2 \operatorname{Re}(a_{jj}) \\ &= 0. \end{aligned}$$

Therefore, every diagonal entry of a skew-Hermitian matrix must be purely imaginary (or zero). For example, $a_{jj} = 3i$ works, since

$$\begin{aligned} -\overline{3i} &= -(-3i) \\ &= 3i, \end{aligned}$$

but $a_{jj} = 3$ does not, since $-\overline{3} = -3$.

Example 0.7 (A skew-Hermitian matrix). The matrix

$$\mathbf{A} = \begin{bmatrix} 2i & 1+i \\ -1+i & 0 \end{bmatrix}$$

is skew-Hermitian: transposing and conjugating gives

$$\begin{aligned} \mathbf{A}^H &= \begin{bmatrix} \overline{2i} & \overline{-1+i} \\ \overline{-1+i} & \overline{0} \end{bmatrix} \\ &= \begin{bmatrix} -2i & -1-i \\ 1-i & 0 \end{bmatrix} \\ &= -\mathbf{A}. \end{aligned}$$

Its diagonal entries $2i$ and 0 are purely imaginary.

Theorem 0.4 (Hermitian and skew-Hermitian decomposition). *Every square complex matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ can be uniquely expressed as the sum of a Hermitian matrix $\mathbf{H}$ (Definition 0.5) and a skew-Hermitian matrix $\mathbf{S}$ (Definition 0.6):*

$$\mathbf{A} = \mathbf{H} + \mathbf{S},$$

where

$$\mathbf{H} = \frac{\mathbf{A} + \mathbf{A}^H}{2} \quad \text{and} \quad \mathbf{S} = \frac{\mathbf{A} - \mathbf{A}^H}{2}.$$

Proof

Proof. **Existence:** Define $\mathbf{H} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^H)$ and $\mathbf{S} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^H)$. Their sum is:

$$\begin{aligned}
\mathbf{H} + \mathbf{S} &= \frac{\mathbf{A} + \mathbf{A}^H}{2} + \frac{\mathbf{A} - \mathbf{A}^H}{2} \\
&= \frac{2\mathbf{A}}{2} \\
&= \mathbf{A}.
\end{aligned}$$

Applying the properties of the Hermitian adjoint (Theorem 0.1):

$$\begin{aligned}
\mathbf{H}^H &= \left(\frac{\mathbf{A} + \mathbf{A}^H}{2} \right)^H \\
&= \frac{\mathbf{A}^H + (\mathbf{A}^H)^H}{2} \\
&= \frac{\mathbf{A}^H + \mathbf{A}}{2} \\
&= \mathbf{H}, \\
\mathbf{S}^H &= \left(\frac{\mathbf{A} - \mathbf{A}^H}{2} \right)^H \\
&= \frac{\mathbf{A}^H - (\mathbf{A}^H)^H}{2} \\
&= \frac{\mathbf{A}^H - \mathbf{A}}{2} \\
&= -\mathbf{S}.
\end{aligned}$$

Thus $\mathbf{H}$ is Hermitian and $\mathbf{S}$ is skew-Hermitian.

Uniqueness: Suppose $\mathbf{A} = \mathbf{H}' + \mathbf{S}'$ with $(\mathbf{H}')^H = \mathbf{H}'$ and $(\mathbf{S}')^H = -\mathbf{S}'$. Taking the Hermitian adjoint of both sides yields:

$$\begin{aligned}
\mathbf{A}^H &= (\mathbf{H}' + \mathbf{S}')^H \\
&= (\mathbf{H}')^H + (\mathbf{S}')^H \\
&= \mathbf{H}' - \mathbf{S}'.
\end{aligned}$$

Adding $\mathbf{A} = \mathbf{H}' + \mathbf{S}'$ and $\mathbf{A}^H = \mathbf{H}' - \mathbf{S}'$ gives $\mathbf{A} + \mathbf{A}^H = 2\mathbf{H}'$, so

$$\begin{aligned}
\mathbf{H}' &= \frac{1}{2}(\mathbf{A} + \mathbf{A}^H) \\
&= \mathbf{H}.
\end{aligned}$$

Subtracting the two equations gives $\mathbf{A} - \mathbf{A}^H = 2\mathbf{S}'$, so

$$\begin{aligned}
\mathbf{S}' &= \frac{1}{2}(\mathbf{A} - \mathbf{A}^H) \\
&= \mathbf{S}.
\end{aligned}$$

□

- [1] Wikipedia contributors, “Conjugate transpose — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Conjugate_transpose
- [2] S. Axler, *Linear algebra done right*, 4th ed. in Undergraduate texts in mathematics. Springer, 2024. doi: [10.1007/978-3-031-41026-0](https://doi.org/10.1007/978-3-031-41026-0).
- [3] Wikipedia contributors, “Hermitian matrix — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Hermitian_matrix
- [4] Wikipedia contributors, “Skew-hermitian matrix — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Skew-Hermitian_matrix