Column Space and Null Space

Last modified: 2026-11-24 09:19:45 (PST)

Column space and null space

Definition 0.1 (Column space, range, image)**.** The **column space** [1] of an $m \times n$ matrix $\mathbf{A}$ is

$$\mathcal{C}(\mathbf{A}) \stackrel{\text{def}}{=} \{\mathbf{A}\underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^n\},$$

the set of all vectors $\mathbf{A}\underset{\sim}{x}$ in $\mathbb{R}^m$. It is also called the **range** or **image** of $\mathbf{A}$.

Definition 0.2 (Row space)**.** The **row space** of an $m \times n$ matrix $\mathbf{A}$ is $\mathcal{C}(\mathbf{A}^\top)$, a set of vectors in $\mathbb{R}^n$.

Example 0.1 (The column space of a rank-one matrix)**.** Let

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -2 \\ 3 & -6 & -6 \end{bmatrix}.$$

For any $\underset{\sim}{x} \in \mathbb{R}^3$,

$$\begin{aligned} \mathbf{A}\underset{\sim}{x} &= (x_1 - 2x_2 - 2x_3,\ 3\,(x_1 - 2x_2 - 2x_3)) \\ &= (x_1 - 2x_2 - 2x_3)\,(1, 3) \end{aligned}$$

(Definition 8 in Matrix Multiplication). The number $x_1 - 2x_2 - 2x_3$ can be any real number (take $x_2 = 0$ and $x_3 = 0$), so $\mathcal{C}(\mathbf{A}) = \{c\,(1, 3) : c \in \mathbb{R}\}$, a line in $\mathbb{R}^2$. The vector $(1, 0)$ is not in $\mathcal{C}(\mathbf{A})$: $c\,(1, 3)$ has second entry $3c$, which is 0 only if $c = 0$, and then the first entry is 0, not 1.

Theorem 0.1 (The column space is the span of the columns)**.** *If* $\mathbf{A}$ *is an* $m \times n$ *matrix with columns* $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$*, then*

$$\mathcal{C}(\mathbf{A}) = \operatorname{span}\{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n\},$$

so $\mathcal{C}(\mathbf{A})$ is a subspace of $\mathbb{R}^m$, and the row space $\mathcal{C}(\mathbf{A}^\top)$ is a subspace of $\mathbb{R}^n$.

Proof

Proof. By Theorem 3 in Subspaces and Rank, $\mathbf{A}\underset{\sim}{x} = x_1\underset{\sim}{a}_1 + \cdots + x_n\underset{\sim}{a}_n$, and as $\underset{\sim}{x}$ ranges over $\mathbb{R}^n$, the coefficients $x_1, \ldots, x_n$ range over all lists of n numbers. So the set of all $\mathbf{A}\underset{\sim}{x}$ (Definition 0.1) is the set of all linear combinations of the columns, which is their span (Definition 5 in Subspaces and Rank). A span is a subspace (Theorem 2 in Subspaces and Rank). Applying the same argument to $\mathbf{A}^\top$, whose columns are the rows of $\mathbf{A}$, shows the row space is a subspace of $\mathbb{R}^n$. □

Example 0.2 (Reading off the column space and row space from the columns and rows)**.** For $\mathbf{A}$ in Example 0.1, the columns are $(1, 3)$, $(-2, -6) = -2\,(1, 3)$ and $(-2, -6) = -2\,(1, 3)$, so by Theorem 0.1 $\mathcal{C}(\mathbf{A}) = \operatorname{span}\{(1, 3)\}$, the same line found in Example 0.1. The rows are $(1, -2, -2)$ and $(3, -6, -6) = 3\,(1, -2, -2)$, so the row space is $\operatorname{span}\{(1, -2, -2)\}$, a line in $\mathbb{R}^3$.

Definition 0.3 (Null space, kernel)**.** The **null space** of an $m \times n$ matrix $\mathbf{A}$ is

$$\mathcal{N}(\mathbf{A}) \stackrel{\text{def}}{=} \{\underset{\sim}{x} \in \mathbb{R}^n : \mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m\},$$

the set of solutions of the linear system $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m$. It is also called the **kernel** of $\mathbf{A}$.

Definition 0.4 (Left null space)**.** The **left null space** of an $m \times n$ matrix $\mathbf{A}$ is $\mathcal{N}(\mathbf{A}^\top)$, a set of vectors in $\mathbb{R}^m$.

Example 0.3 (The null space and left null space of a rank-one matrix)**.** For $\mathbf{A}$ in Example 0.1, $\mathbf{A}\underset{\sim}{x} = (x_1 - 2x_2 - 2x_3,\ 3\,(x_1 - 2x_2 - 2x_3))$, so $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_2$ exactly when $x_1 = 2x_2 + 2x_3$. Writing

$$\begin{aligned}\underset{\sim}{x} &= (2x_2 + 2x_3, x_2, x_3)\\ &= x_2\,(2, 1, 0) + x_3\,(2, 0, 1)\end{aligned}$$

shows $\mathcal{N}(\mathbf{A}) = \operatorname{span}\{(2, 1, 0), (2, 0, 1)\}$, a plane in $\mathbb{R}^3$.
For the left null space, $\mathbf{A}^\top y = (y_1 + 3y_2,\ -2\,(y_1 + 3y_2),\ -2\,(y_1 + 3y_2))$, which is $\underset{\sim}{0}_3$ exactly when $y_1 = -3y_2$, so $\mathcal{N}(\mathbf{A}^\top) = \operatorname{span}\{(-3, 1)\}$, a line in $\mathbb{R}^2$.

Exercise 0.1 (Testing whether a vector is in a column space)**.** A vector $\underset{\sim}{b}$ is in the column space of a matrix $\mathbf{A}$ exactly when some $\underset{\sim}{x}$ gives $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \qquad \underset{\sim}{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

Decide whether $\underset{\sim}{b}$ is in $\mathcal{C}(\mathbf{A})$.

Hint

Try $\underset{\sim}{x} = (x_1, 0)$ and solve for x_1.

Solution

Solution 0.1. Take $\underset{\sim}{x} = (2, 0)$. Then

$$\mathbf{A}\underset{\sim}{x} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 0 \\ 3 \cdot 2 + 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \underset{\sim}{b},$$

so $\underset{\sim}{b}$ is in $\mathcal{C}(\mathbf{A})$.

Exercise 0.2 (Testing whether a vector is in a row space)**.** The row space of a matrix $\mathbf{A}$ is the column space of its transpose. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

and let $\underset{\sim}{v}$ be the column vector $(1, 2)$.
Decide whether $\underset{\sim}{v}$ is in the row space $\mathcal{C}(\mathbf{A}^\top)$.

Hint

Write out $\mathbf{A}^\top$ and look for $\underset{\sim}{u}$ with $\mathbf{A}^\top\underset{\sim}{u} = \underset{\sim}{v}$.

Solution

Solution 0.2. Here

$$\mathbf{A}^\top = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

Take $\underset{\sim}{u} = (1, 0)$. Then

$$\mathbf{A}^\top\underset{\sim}{u} = \begin{bmatrix} 1 \cdot 1 + 3 \cdot 0 \\ 2 \cdot 1 + 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underset{\sim}{v},$$

so $\underset{\sim}{v}$ is in the row space.

Exercise 0.3 (Showing that a column space is all of $\mathbb{R}^2$)**.** Theorem 0.1 describes a column space as the span of the columns of $\mathbf{A}$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Show that $\mathcal{C}(\mathbf{A}) = \mathbb{R}^2$.

Hint

Two of the columns of $\mathbf{A}$ are the indicator vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$.

Solution

Solution 0.3. By Theorem 0.1, $\mathcal{C}(\mathbf{A}) = \text{span}\{(1,0),(0,1),(1,1)\}$. Every $(b_1, b_2) \in \mathbb{R}^2$ is $b_1\,(1,0) + b_2\,(0,1) + 0\,(1,1)$, so it is in that span. The column space also lies inside $\mathbb{R}^2$, so $\mathcal{C}(\mathbf{A}) = \mathbb{R}^2$.

Exercise 0.4 (Finding a null space)**.** The null space of a matrix $\mathbf{A}$ holds every $\underset{\sim}{x}$ with $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find $\mathcal{N}(\mathbf{A})$ as a span.

Hint

Write out the two entries of $\mathbf{A}\underset{\sim}{x}$ and make each 0.

Solution

Solution 0.4. For $\underset{\sim}{x} = (x_1, x_2, x_3)$,

$$\mathbf{A}\underset{\sim}{x} = \begin{bmatrix} x_1 + 2x_2 \\ x_3 \end{bmatrix}.$$

This vector is $\underset{\sim}{0}_2$ exactly when $x_3 = 0$ and $x_1 = -2x_2$. So $\underset{\sim}{x} = (-2x_2, x_2, 0) = x_2\,(-2,1,0)$, and

$$\mathcal{N}(\mathbf{A}) = \text{span}\{(-2,1,0)\}.$$

Exercise 0.5 (Testing whether a vector is in a left null space)**.** The left null space of a matrix $\mathbf{A}$ is the null space of its transpose. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

and let y be the column vector $(2, -1)$.
Decide whether $\underset{\sim}{y}$ is in $\mathcal{N}(\mathbf{A}^\top)$.

Hint

Compute $\mathbf{A}^\top \underset{\sim}{y}$.

Solution

Solution 0.5. Here

$$\mathbf{A}^\top \underset{\sim}{y} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 4-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so $\underset{\sim}{y}$ is in $\mathcal{N}(\mathbf{A}^\top)$. Here $\mathbf{A}$ equals its own transpose, so $\underset{\sim}{y}$ is in $\mathcal{N}(\mathbf{A})$ as well.

Theorem 0.2 (A null space is a subspace)**.** *For any $m \times n$ matrix $\mathbf{A}$, $\mathcal{N}(\mathbf{A})$ (Definition 0.3) is a subspace of $\mathbb{R}^n$ (Definition 4 in Subspaces and Rank).*

Proof

Proof. $\mathbf{A}\underset{\sim}{0}_n = \underset{\sim}{0}_m$, so $\underset{\sim}{0}_n \in \mathcal{N}(\mathbf{A})$, and the null space is not empty. If $\underset{\sim}{u}, \underset{\sim}{w} \in \mathcal{N}(\mathbf{A})$ and $c \in \mathbb{R}$, then

$$\begin{aligned} \mathbf{A}\left(\underset{\sim}{u} + \underset{\sim}{w}\right) &= \mathbf{A}\underset{\sim}{u} + \mathbf{A}\underset{\sim}{w} && \text{(Theorem 2 in Matrix Multiplication)} \\ &= \underset{\sim}{0}_m + \underset{\sim}{0}_m && (\underset{\sim}{u} \text{ and } \underset{\sim}{w} \text{ are in the null space}) \\ &= \underset{\sim}{0}_m && (\text{adding } \underset{\sim}{0}_m \text{ changes nothing}) \end{aligned}$$

and, for each entry $i = 1, \ldots, m$,

$$\begin{aligned} \left(\mathbf{A}\left(c\underset{\sim}{u}\right)\right)_i &= a_{i1}(cu_1) + \cdots + a_{in}(cu_n) && \text{(Definition 8 in Matrix Multiplication)} \\ &= c\left(a_{i1}u_1 + \cdots + a_{in}u_n\right) && (\text{factor } c \text{ out of each term}) \\ &= c\left(\mathbf{A}\underset{\sim}{u}\right)_i && \text{(Definition 8 in Matrix Multiplication)} \\ &= c \cdot 0 && (\underset{\sim}{u} \text{ is in the null space}) \\ &= 0, && (\text{arithmetic}) \end{aligned}$$

so $\mathbf{A}\left(c\underset{\sim}{u}\right) = \underset{\sim}{0}_m$. Therefore $\underset{\sim}{u} + \underset{\sim}{w}$ and $c\underset{\sim}{u}$ are in $\mathcal{N}(\mathbf{A})$. □

Example 0.4 (The solutions of $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}$ with $\underset{\sim}{b} \neq \underset{\sim}{0}$ are not a subspace)**.** In Example 0.3, $(2,1,0)$ and $(2,0,1)$ are in $\mathcal{N}(\mathbf{A})$, and so is their sum $(4,1,1)$, since $4-2\cdot 1-2\cdot 1 = 0$. By contrast, the solutions of $\mathbf{A}\underset{\sim}{x} = (1,3)$ for the same $\mathbf{A}$ are the vectors with $x_1 - 2x_2 - 2x_3 = 1$. That set does not contain $\underset{\sim}{0}_3$, so by Theorem 1 in Subspaces and Rank it is not a subspace.

Exercise 0.6 (Showing that the solutions of two equations are a subspace)**.** Theorem 0.2 turns a question about a subspace into a question about a null space. Let $\mathcal{S}$ hold the solutions $\underset{\sim}{x} \in \mathbb{R}^3$ of the two equations $x_1 + x_2 - x_3 = 0$ and $x_2 = 2x_3$.
Show that $\mathcal{S}$ is a subspace of $\mathbb{R}^3$.

Hint

Write both equations as $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_2$ for one 2×3 matrix $\mathbf{A}$.

Solution

Solution 0.6. Move every term to the left side: $x_1 + x_2 - x_3 = 0$ and $x_2 - 2x_3 = 0$. With

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix},$$

the two equations say $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_2$, so $\mathcal{S} = \mathcal{N}(\mathbf{A})$. By Theorem 0.2, $\mathcal{S}$ is a subspace of $\mathbb{R}^3$.

Theorem 0.3 (Independent columns exactly when the null space is trivial)**.** *For an $m \times n$ matrix $\mathbf{A}$, the following are equivalent:*

1. *The columns of $\mathbf{A}$ are linearly independent (Definition 1 in Subspaces and Rank).*
2. *$\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m$ only for $\underset{\sim}{x} = \underset{\sim}{0}_n$.*
3. *$\mathcal{N}(\mathbf{A}) = \{\underset{\sim}{0}_n\}$ (Definition 0.3).*

So $\mathbf{A}$ has full column rank (Definition 3 in Subspaces and Rank) exactly when its null space contains only the zero vector.

Proof

Proof. Write $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ for the columns of $\mathbf{A}$. For $\underset{\sim}{x} = (x_1, \ldots, x_n)$, the matrix-vector product is the combination of the columns with weights $x_1, \ldots, x_n$:

$$\mathbf{A}\underset{\sim}{x} = x_1\underset{\sim}{a}_1 + \cdots + x_n\underset{\sim}{a}_n.$$

So $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m$ says exactly that $x_1\underset{\sim}{a}_1 + \cdots + x_n\underset{\sim}{a}_n = \underset{\sim}{0}_m$. By Definition 1 in Subspaces

and Rank, the columns are linearly independent exactly when the only weights with this property are $x_1 = \cdots = x_n = 0$. That is statement 2, so statements 1 and 2 are equivalent. Statement 2 says that the only vector in $\mathcal{N}(\mathbf{A})$ is $\underset{\sim}{0}_n$, which is statement 3. The last sentence is the definition of full column rank (Definition 3 in Subspaces and Rank). □

Example 0.5 (Checking independence through the null space)**.** For $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$,

$$\mathbf{A}\underset{\sim}{x} = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$

This is $\underset{\sim}{0}_3$ only if $x_2 = 0$ (subtract the first entry from the second), and then $x_1 = 0$. So $\mathcal{N}(\mathbf{A}) = \{\underset{\sim}{0}_2\}$ and the two columns are linearly independent.

For $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$, $\mathbf{B}(2, -1) = \underset{\sim}{0}_3$, so $\mathcal{N}(\mathbf{B})$ contains a nonzero vector and the columns of $\mathbf{B}$ are linearly dependent.

Exercise 0.7 (Testing independent columns through the null space)**.** Theorem 0.3 turns a question about independence into one about solving $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 1 \end{bmatrix}.$$

Decide whether the columns of $\mathbf{A}$ are linearly independent.

Hint

Solve $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_3$, starting with the third entry.

Solution

Solution 0.7. For $\underset{\sim}{x} = (x_1, x_2)$,

$$\mathbf{A}\underset{\sim}{x} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \\ x_2 \end{bmatrix}.$$

The third entry gives $x_2 = 0$, and then the first gives $x_1 = 0$. So $\mathcal{N}(\mathbf{A}) = \{\underset{\sim}{0}_2\}$, and by Theorem 0.3 the columns of $\mathbf{A}$ are linearly independent.

Theorem 0.4 (The row space and null space share only the zero vector)**.** *For any $m \times n$ matrix* $\mathbf{A}$*,*

$$\mathcal{C}(\mathbf{A}^\top) \cap \mathcal{N}(\mathbf{A}) = \{\underset{\sim}{0}_n\}.$$

Proof

Proof. Both sets are subspaces, by Theorem 0.1 and Theorem 0.2, so both contain $\underset{\sim}{0}_n$ (Theorem 1 in Subspaces and Rank).
Now take any $\underset{\sim}{x}$ in both sets. Being in the row space means $\underset{\sim}{x} = \mathbf{A}^\top \underset{\sim}{u}$ for some $\underset{\sim}{u} \in \mathbb{R}^m$ (Definition 0.1), and being in the null space means $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m$ (Definition 0.3). Then

$$\begin{aligned}
\underset{\sim}{x}^\top \underset{\sim}{x} &= (\mathbf{A}^\top \underset{\sim}{u})^\top \underset{\sim}{x} && \text{(substitute } \underset{\sim}{x} = \mathbf{A}^\top \underset{\sim}{u} \text{ in the first factor)} \\
&= \left(\underset{\sim}{u}^\top (\mathbf{A}^\top)^\top\right) \underset{\sim}{x} && \text{(Theorem 6 in Matrix Multiplication)} \\
&= (\underset{\sim}{u}^\top \mathbf{A})\, \underset{\sim}{x} && \text{(transposing twice returns the original matrix)} \\
&= \underset{\sim}{u}^\top (\mathbf{A}\underset{\sim}{x}) && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \underset{\sim}{u}^\top \underset{\sim}{0}_m && \text{(substitute } \mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m) \\
&= 0. && \text{(every term of the inner product is 0)}
\end{aligned}$$

But $\underset{\sim}{x}^\top \underset{\sim}{x} = x_1^2 + \cdots + x_n^2$, which is 0 only when every $x_i = 0$. So $\underset{\sim}{x} = \underset{\sim}{0}_n$. □

Example 0.6 (The row space and null space in Example 0.3)**.** For $\mathbf{A}$ in Example 0.1, the row space is $\text{span}\{(1, -2, -2)\}$ and the null space is $\text{span}\{(2, 1, 0), (2, 0, 1)\}$ (Example 0.3). A vector $c\,(1, -2, -2)$ of the row space is in the null space only if

$$\begin{aligned}
\mathbf{A}\,(c\,(1, -2, -2)) &= c\,(1 - 2 \cdot (-2) - 2 \cdot (-2),\ 3\,(1 - 2 \cdot (-2) - 2 \cdot (-2))) && \text{(Example 0.3's formula for } \mathbf{A}\underset{\sim}{x}) \\
&= c\,(9, 27) && \text{(arithmetic)}
\end{aligned}$$

is $\underset{\sim}{0}_2$, that is, only if $c = 0$. So the two subspaces share only $\underset{\sim}{0}_3$.

Exercise 0.8 (Testing a row-space vector against the null space)**.** Theorem 0.4 says that a nonzero vector in the row space is never in the null space. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \underset{\sim}{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

so that $\underset{\sim}{v} = (1, 1, 0) + (0, 1, 1)$ adds the two rows of $\mathbf{A}$.
Check that $\underset{\sim}{v}$ is not in $\mathcal{N}(\mathbf{A})$.

Hint

Multiply each row of $\mathbf{A}$ by $\underset{\sim}{v}$.

Solution

Solution 0.8. Here

$$\mathbf{A}\underset{\sim}{v} = \begin{bmatrix} 1+2+0 \\ 0+2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \neq \underset{\sim}{0}_2,$$

so $\underset{\sim}{v}$ is not in $\mathcal{N}(\mathbf{A})$, as Theorem 0.4 requires of a nonzero vector in the row space.

Definition 0.5 (Cross-product matrix)**.** Let $\mathbf{A}$ be an $m \times n$ matrix with columns $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ and $\mathbf{B}$ an $m \times q$ matrix with columns $\underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_q$. The **cross-product matrix** of $\mathbf{A}$ and $\mathbf{B}$ is the $n \times q$ matrix $\mathbf{A}^\top\mathbf{B}$, whose entry (i, j) is the dot product $\underset{\sim}{a}_i \cdot \underset{\sim}{b}_j$.

This is what R's `crossprod(A, B)` computes, and `crossprod(A)` is $\mathbf{A}^\top\mathbf{A}$. In NumPy it has no name of its own: write `A.T @ B`.

For two column vectors it is the 1×1 matrix holding their dot product. It is unrelated to the cross product $\underset{\sim}{u} \times \underset{\sim}{v}$ of two vectors in $\mathbb{R}^3$, and differs from the Frobenius inner product $\operatorname{tr}(\mathbf{A}^\top\mathbf{B})$, which sums the diagonal of this matrix into a single number.

Example 0.7 (A cross-product matrix)**.** Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then $\mathbf{A}^\top\mathbf{B}$ is 2×1, and

$$\begin{aligned} \mathbf{A}^\top\mathbf{B} &= \begin{bmatrix} \underset{\sim}{a}_1 \cdot \underset{\sim}{b}_1 \\ \underset{\sim}{a}_2 \cdot \underset{\sim}{b}_1 \end{bmatrix} && \text{(Definition 0.5)} \\ &= \begin{bmatrix} 1+2+0 \\ 0+2+3 \end{bmatrix} && \text{(multiply and add)} \\ &= \begin{bmatrix} 3 \\ 5 \end{bmatrix}. && \text{(add)} \end{aligned}$$

Exercise 0.9 (Computing a cross-product matrix)**.** In R, `crossprod(A, B)` computes the cross-product matrix $\mathbf{A}^\top\mathbf{B}$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Compute $\mathbf{A}^\top \mathbf{B}$.

Hint

Entry i is the dot product of column i of $\mathbf{A}$ with $(2, 3)$.

Solution

Solution 0.9. The columns of $\mathbf{A}$ are $(1, 1)$ and $(0, 1)$, so

$$\mathbf{A}^\top \mathbf{B} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 3 \\ 0 \cdot 2 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Remark (Column-column products)

Remark 0.1 (Column-column products). Ordinary matrix multiplication pairs a row of the left factor with a column of the right factor, which makes each entry a row-by-column product [2, Sec. 2]. Each entry of a cross-product matrix instead pairs a column of $\mathbf{A}$ with a column of $\mathbf{B}$, so in these notes we call it a column-column product.

Definition 0.6 (Gram matrix)**.** The **Gram matrix** [3] of an $m \times n$ matrix $\mathbf{A}$ is the $n \times n$ matrix $\mathbf{A}^\top \mathbf{A}$, the cross-product matrix of $\mathbf{A}$ with itself.

Example 0.8 (A Gram matrix and its entries)**.** Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, with columns $\underset{\sim}{a}_1 = (1, 1, 0)$ and $\underset{\sim}{a}_2 = (0, 1, 1)$. Then

$$\begin{aligned} \mathbf{A}^\top \mathbf{A} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} && \text{(Definition 6 in Matrices)} \\ &= \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 \end{bmatrix} && \text{(Definition 1 in Matrix Multiplication)} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. && \text{(multiply and add)} \end{aligned}$$

Row i of $\mathbf{A}^\top$ is column i of $\mathbf{A}$, so entry (i, j) of the Gram matrix is the dot product $\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j$: for instance,

$$\begin{aligned} \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2 &= 0 + 1 + 0 \\ &= 1 \end{aligned}$$

is entry $(1, 2)$.

Exercise 0.10 (Computing a Gram matrix)**.** The Gram matrix holds the dot products of every pair of columns of $\mathbf{A}$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Compute $\mathbf{A}^\top \mathbf{A}$.

Hint

Entry (i, j) is $\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j$, so entries $(1, 2)$ and $(2, 1)$ agree.

Solution

Solution 0.10. The columns of $\mathbf{A}$ are $\underset{\sim}{a}_1 = (1, 0, 1)$ and $\underset{\sim}{a}_2 = (2, 1, 0)$, so

$$\begin{aligned} \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1 &= 1 + 0 + 1 = 2, \\ \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2 &= 2 + 0 + 0 = 2, \\ \underset{\sim}{a}_2 \cdot \underset{\sim}{a}_2 &= 4 + 1 + 0 = 5, \end{aligned}$$

and

$$\mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}.$$

Theorem 0.5 (The cross-product matrix is the sum of the outer products of the rows)**.** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix with rows* $\underset{\sim}{r}_1^\top, \ldots, \underset{\sim}{r}_m^\top$*. Let* $\mathbf{B}$ *be an* $m \times q$ *matrix with rows* $\underset{\sim}{s}_1^\top, \ldots, \underset{\sim}{s}_m^\top$*. Then the cross-product matrix of* $\mathbf{A}$ *with* $\mathbf{B}$ *is*

$$\mathbf{A}^\top \mathbf{B} = \sum_{i=1}^{m} \underset{\sim}{r}_i \, \underset{\sim}{s}_i^\top.$$

Proof

Proof. Compare entry (j,k) of each side.

$$
\begin{aligned}
(\mathbf{A}^\top\mathbf{B})_{jk} &= \sum_{i=1}^{m}(\mathbf{A})_{ij}(\mathbf{B})_{ik} && \text{(Definition 1 in Matrix Multiplication)}\\
&= \sum_{i=1}^{m}(\underset{\sim}{r}_i)_j(\underset{\sim}{s}_i)_k && \text{(entries of the rows)}\\
&= \sum_{i=1}^{m}(\underset{\sim}{r}_i\,\underset{\sim}{s}_i{}^\top)_{jk} && \text{(Definition 3 in Orthonormal Bases, Outer Products and Hilbert Spaces)}\\
&= \left(\sum_{i=1}^{m}\underset{\sim}{r}_i\,\underset{\sim}{s}_i{}^\top\right)_{jk}. && \text{(entries of a sum)}
\end{aligned}
$$

Every entry agrees, so the two matrices agree. □

Example 0.9 (The Gram matrix and a cross-product as sums of outer products)**.** Let $\mathbf{A} = \begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}$, $\mathbf{B} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ be the matrices of Example 0.7. The rows of $\mathbf{A}$ are $\underset{\sim}{r}_1 = (1,0)$, $\underset{\sim}{r}_2 = (1,1)$, $\underset{\sim}{r}_3 = (0,1)$. Taking $\mathbf{B} = \mathbf{A}$ in Theorem 0.5, the Gram matrix is the sum of the outer products of the rows with themselves:

$$
\begin{aligned}
\mathbf{A}^\top\mathbf{A} &= \underset{\sim}{r}_1\,\underset{\sim}{r}_1{}^\top + \underset{\sim}{r}_2\,\underset{\sim}{r}_2{}^\top + \underset{\sim}{r}_3\,\underset{\sim}{r}_3{}^\top && \text{(Theorem 0.5)}\\
&= \begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}1&1\\1&1\end{bmatrix} + \begin{bmatrix}0&0\\0&1\end{bmatrix} && \text{(Definition 3 in Orthonormal Bases, Outer Products and Hilbert Spaces}\\
&= \begin{bmatrix}2&1\\1&2\end{bmatrix}. && \text{(add)}
\end{aligned}
$$

This is the Gram matrix of Example 0.8. Now keep the column $\mathbf{B}$ above. Its rows are the 1-vectors $\underset{\sim}{s}_1 = (1)$, $\underset{\sim}{s}_2 = (2)$, $\underset{\sim}{s}_3 = (3)$, so

$$
\begin{aligned}
\mathbf{A}^\top\mathbf{B} &= 1\underset{\sim}{r}_1 + 2\underset{\sim}{r}_2 + 3\underset{\sim}{r}_3 && \text{(Theorem 0.5)}\\
&= \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix} && \text{(scalar multiples)}\\
&= \begin{bmatrix}3\\5\end{bmatrix}, && \text{(add)}
\end{aligned}
$$

the same as in Example 0.7.

Exercise 0.11 (Computing a cross-product matrix one row at a time)**.** Theorem 0.5 builds a cross-product matrix by adding one outer product per row, which is how it can be updated as observations arrive. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Compute $\mathbf{A}^\top \mathbf{B}$ by adding two outer products.

Hint

Rows of $\mathbf{A}$ are $(1, 2)$ and $(0, 1)$, and rows of $\mathbf{B}$ are 3 and 1.

Solution

Solution 0.11. By Theorem 0.5,

$$\mathbf{A}^\top \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [3] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Multiplying directly gives the same: $\mathbf{A}^\top \mathbf{B} = (1 \cdot 3 + 0 \cdot 1,\ 2 \cdot 3 + 1 \cdot 1) = (3, 7)$.

Theorem 0.6 ($\mathbf{A}^\top \mathbf{A}$ has the same null space as $\mathbf{A}$)**.** *For any $m \times n$ matrix $\mathbf{A}$,*

$$\mathcal{N}(\mathbf{A}^\top \mathbf{A}) = \mathcal{N}(\mathbf{A}).$$

Proof

Proof. $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A}^\top \mathbf{A})$**.** If $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m$, then

$$\begin{aligned} \mathbf{A}^\top \mathbf{A}\underset{\sim}{x} &= \mathbf{A}^\top \underset{\sim}{0}_m \\ &= \underset{\sim}{0}_n. \end{aligned}$$

$\mathcal{N}(\mathbf{A}^\top \mathbf{A}) \subseteq \mathcal{N}(\mathbf{A})$**.** If $\mathbf{A}^\top \mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_n$, then

$$\begin{aligned} 0 &= \underset{\sim}{x}^\top \underset{\sim}{0}_n && \text{(every term of the inner product is 0)} \\ &= \underset{\sim}{x}^\top (\mathbf{A}^\top \mathbf{A}\underset{\sim}{x}) && \text{(substitute } \underset{\sim}{0}_n = \mathbf{A}^\top \mathbf{A}\underset{\sim}{x}\text{)} \\ &= (\underset{\sim}{x}^\top \mathbf{A}^\top)(\mathbf{A}\underset{\sim}{x}) && \text{(Theorem 1 in Matrix Multiplication)} \\ &= (\mathbf{A}\underset{\sim}{x})^\top (\mathbf{A}\underset{\sim}{x}) && \text{(Theorem 6 in Matrix Multiplication)} \\ &= (\mathbf{A}\underset{\sim}{x}) \cdot (\mathbf{A}\underset{\sim}{x}) && \text{(Example 3 in Matrix Multiplication)} \\ &= \|\mathbf{A}\underset{\sim}{x}\|^2. && \text{(square both sides of Equation 5 in Vectors)} \end{aligned}$$

A vector whose Euclidean norm is 0 has every entry 0, so $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_m$. □

Example 0.10 (The null space of a Gram matrix)**.** For $\mathbf{A}$ in Example 0.1,

$$\mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & -6 \\ -2 & -6 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 3 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}.$$

Every row is a multiple of $(1, -2, -2)$, so $\mathbf{A}^\top \mathbf{A} \underset{\sim}{x} = \underset{\sim}{0}_3$ exactly when $x_1 - 2x_2 - 2x_3 = 0$, the same condition as for $\mathcal{N}(\mathbf{A})$ in Example 0.3.

Exercise 0.12 (Finding the null space of a Gram matrix)**.** Theorem 0.6 lets the null space of a Gram matrix be read off from the original matrix. Let $\mathbf{A}$ be the 3×2 matrix whose every entry is 1.
Find $\mathcal{N}(\mathbf{A}^\top \mathbf{A})$.

Hint

Find $\mathcal{N}(\mathbf{A})$ first. Each entry of $\mathbf{A}\underset{\sim}{x}$ is $x_1 + x_2$.

Solution

Solution 0.12. Each entry of $\mathbf{A}\underset{\sim}{x}$ is $x_1 + x_2$, so $\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}_3$ exactly when $x_2 = -x_1$, and $\mathcal{N}(\mathbf{A}) = \operatorname{span}\{(1, -1)\}$. By Theorem 0.6,

$$\mathcal{N}(\mathbf{A}^\top \mathbf{A}) = \operatorname{span}\{(1, -1)\}.$$

As a check, $\mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$, which sends $(1, -1)$ to $\underset{\sim}{0}_2$.

[1] Wikipedia contributors, "Row and column spaces — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Row_and_column_spaces

[2] "Chapter I: Linear algebra, basic notions." Lecture notes, Department of Mathematics, Northwestern University. Available: https://sites.math.northwestern.edu/len/LinAlg/chap1.pdf

[3] Wikipedia contributors, "Gram matrix — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Gram_matrix