Bases and Dimension

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Bases and dimension

Definition 0.1 (Basis)**.** A **basis** [1] of a subspace $\mathcal{S}$ (Definition 4 in Subspaces and Rank) is a list of vectors that are linearly independent (Definition 1 in Subspaces and Rank) and span $\mathcal{S}$ (Definition 5 in Subspaces and Rank).

Example 0.1 (Two bases of $\mathbb{R}^2$)**.**

- $(1,0)$ and $(0,1)$ form a basis of $\mathbb{R}^2$: they are linearly independent, and any $(x,y) \in \mathbb{R}^2$ equals $x\,(1,0) + y\,(0,1)$.
- $(1,0)$ and $(1,1)$ also form a basis of $\mathbb{R}^2$: Example 1 in Subspaces and Rank shows they are linearly independent, and any (x,y) equals $(x-y)\,(1,0) + y\,(1,1)$, because $(x-y)+y = x$ in the first entry and $0+y=y$ in the second.

Example 0.2 (Lists that are not bases of $\mathbb{R}^2$)**.**

- $(1,0)$ and $(2,0)$ are not a basis of $\mathbb{R}^2$. They are not linearly independent, since $2\,(1,0) - (2,0) = \underset{\sim}{0}$, and they do not span $\mathbb{R}^2$: every combination $c_1(1,0) + c_2(2,0)$ has second entry $\widetilde{0}$, so $(0,1)$ is not in their span.
- $(1,0)$, $(0,1)$ and $(1,1)$ span $\mathbb{R}^2$, but they are not a basis: $(1,0) + (0,1) - (1,1) = \underset{\sim}{0}$, so they are not linearly independent.

Exercise 0.1 (Testing whether $(1,1)$ and $(1,-1)$ form a basis of $\mathbb{R}^2$)**.** Describing every vector in $\mathbb{R}^2$ through $(1,1)$ and $(1,-1)$ instead of the axes works only if those two vectors form a basis. Let

$$\underset{\sim}{a}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \qquad \underset{\sim}{a}_2 = \begin{bmatrix}1\\-1\end{bmatrix}.$$

Is $\underset{\sim}{a}_1, \underset{\sim}{a}_2$ a basis of $\mathbb{R}^2$?

Hint

Check both conditions of Definition 0.1. For spanning, look for c_1 and c_2 with $c_1\underset{\sim}{a}_1 + c_2\underset{\sim}{a}_2 = (x, y)^\top$.

Solution

Solution 0.1. Yes. They span $\mathbb{R}^2$: for every $(x, y)^\top \in \mathbb{R}^2$,

$$\frac{x+y}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{x-y}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}.$$

They are linearly independent: if $c_1\underset{\sim}{a}_1 + c_2\underset{\sim}{a}_2 = \underset{\sim}{0}$, then $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$, and adding the two equations gives $c_1 = 0$ and then $c_2 = 0$. By Definition 0.1, $\underset{\sim}{a}_1, \underset{\sim}{a}_2$ is a basis of $\mathbb{R}^2$.

Theorem 0.1 (Coefficients in a basis are unique)**.** *If $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k$ is a basis of a subspace $\mathcal{S}$ (Definition 0.1), then every $\underset{\sim}{x} \in \mathcal{S}$ is a linear combination $\underset{\sim}{x} = c_1\underset{\sim}{a}_1 + \cdots + c_k\underset{\sim}{a}_k$ for exactly one list of numbers $c_1, \ldots, c_k$.*

Proof

Proof. At least one such list exists, because the basis spans $\mathcal{S}$ (Definition 5 in Subspaces and Rank).
Suppose two lists both work:

$$\begin{aligned} c_1\underset{\sim}{a}_1 + \cdots + c_k\underset{\sim}{a}_k &= \underset{\sim}{x} \\ &= d_1\underset{\sim}{a}_1 + \cdots + d_k\underset{\sim}{a}_k. \end{aligned}$$

Subtracting the right side from the left side gives

$$\begin{aligned} \underset{\sim}{0} &= (c_1\underset{\sim}{a}_1 + \cdots + c_k\underset{\sim}{a}_k) - (d_1\underset{\sim}{a}_1 + \cdots + d_k\underset{\sim}{a}_k) && \text{(both combinations equal } \underset{\sim}{x}) \\ &= (c_1 - d_1)\,\underset{\sim}{a}_1 + \cdots + (c_k - d_k)\,\underset{\sim}{a}_k && \text{(regroup, and distribute each } \underset{\sim}{a}_i) \end{aligned}$$

Because $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k$ are linearly independent (Definition 1 in Subspaces and Rank), every coefficient $c_i - d_i$ is 0, so $c_i = d_i$ for each i. □

Example 0.3 (Coefficients of $(3,5)$ in two bases)**.** In the basis $(1,0),(0,1)$ of Example 0.1, $(3,5) = 3\,(1,0) + 5\,(0,1)$, and no other coefficients work. In the basis $(1,0),(1,1)$, the formula in Example 0.1 gives

$$\begin{aligned}(3,5) &= (3-5)\,(1,0) + 5\,(1,1)\\ &= -2\,(1,0) + 5\,(1,1).\end{aligned}$$

The same vector has different coefficients in different bases, but within one basis its coefficients are unique.
The list $(1,0),(0,1),(1,1)$ in Example 0.2 spans $\mathbb{R}^2$ but is not linearly independent, and the coefficients of $(3,5)$ are not unique:

$$\begin{aligned}(3,5) &= 3\,(1,0) + 5\,(0,1) + 0\,(1,1)\\ &= 0\,(1,0) + 2\,(0,1) + 3\,(1,1).\end{aligned}$$

Exercise 0.2 (Counting the ways to write a vector in a basis)**.** The vectors $\underset{\sim}{a}_1 = (1,1)^\top$ and $\underset{\sim}{a}_2 = (1,-1)^\top$ form a basis of $\mathbb{R}^2$, and

$$\begin{bmatrix}5\\1\end{bmatrix} = 3\underset{\sim}{a}_1 + 2\underset{\sim}{a}_2.$$

How many lists of numbers c_1, c_2 satisfy $(5,1)^\top = c_1\underset{\sim}{a}_1 + c_2\underset{\sim}{a}_2$?

Hint

Use Theorem 0.1.

Solution

Solution 0.2. Exactly one, namely $c_1 = 3$ and $c_2 = 2$. By Theorem 0.1, a vector in the span of a basis has only one list of numbers, and $3, 2$ is a list that works.

Definition 0.2 (Coordinates in a basis)**.** Let $\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_k$ be a basis of a subspace $\mathcal{S}$ (Definition 0.1), and let $\underset{\sim}{x} \in \mathcal{S}$. The **coordinates** [2] of $\underset{\sim}{x}$ in this basis are the numbers $c_1, \dots, c_k$ with $\underset{\sim}{x} = c_1\underset{\sim}{a}_1 + \cdots + c_k\underset{\sim}{a}_k$. By Theorem 0.1 there is exactly one such list of numbers.

Definition 0.3 (Coordinate vector)**.** With $\underset{\sim}{x}$ and its coordinates $c_1, \dots, c_k$ as in Definition 0.2, $(c_1, \dots, c_k) \in \mathbb{R}^k$ is the **coordinate vector** of $\underset{\sim}{x}$.

Example 0.4 (Coordinates of $(4,1)$ in two bases of $\mathbb{R}^2$)**.** In the basis $(1,0),(1,1)$ of Example 0.1, the coordinates of $(4,1)$ are 3 and 1:

$$\begin{aligned} 3\,(1,0)+1\,(1,1) &= (3+1,\ 0+1) && \text{(Definition 11 in Vectors, entry by entry)} \\ &= (4,1). && \text{(add)} \end{aligned}$$

In the basis $(1,0),(0,1)$, its coordinates are its entries, 4 and 1.

Exercise 0.3 (Finding coordinates in a basis other than the indicator vectors)**.** The coordinates of a vector depend on the basis, and they match its entries only in the basis of indicator vectors. The vectors $\underset{\sim}{a}_1 = (2,0)^\top$ and $\underset{\sim}{a}_2 = (1,1)^\top$ form a basis of $\mathbb{R}^2$. Let

$$\underset{\sim}{x} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}.$$

Find the coordinates of $\underset{\sim}{x}$ in the basis $\underset{\sim}{a}_1, \underset{\sim}{a}_2$.

Hint

Write $c_1\underset{\sim}{a}_1 + c_2\underset{\sim}{a}_2$ entry by entry, and start from the second entry.

Solution

Solution 0.3. The coordinates of $\underset{\sim}{x}$ are 2 and 3. Entry by entry, $c_1\underset{\sim}{a}_1 + c_2\underset{\sim}{a}_2 = (2c_1+c_2,\ c_2)^\top$. The second entry gives $c_2 = 3$, and then the first gives $2c_1 + 3 = 7$, so $c_1 = 2$. As a check, $2(2,0)^\top + 3(1,1)^\top = (7,3)^\top$.

Exercise 0.4 (Describing a point on a plane in $\mathbb{R}^3$ with two numbers)**.** A point on a plane through the origin in $\mathbb{R}^3$ needs only two numbers, its coordinate vector in a basis of the plane. The vectors $\underset{\sim}{a}_1 = (1,1,0)^\top$ and $\underset{\sim}{a}_2 = (0,1,1)^\top$ form a basis of a plane $\mathcal{S}$, which contains

$$\underset{\sim}{x} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}.$$

Find the coordinate vector of $\underset{\sim}{x}$ in the basis $\underset{\sim}{a}_1, \underset{\sim}{a}_2$.

Hint

Write $c_1\underset{\sim}{a}_1 + c_2\underset{\sim}{a}_2$ entry by entry, then use Definition 0.3.

Solution

Solution 0.4. Entry by entry, $c_1\underset{\sim}{a}_1 + c_2\underset{\sim}{a}_2 = (c_1,\ c_1 + c_2,\ c_2)^\top$. Matching $\underset{\sim}{x}$ gives $c_1 = 2$ and $c_2 = 3$, and the second entry checks: $2 + 3 = 5$. By Definition 0.3, the coordinate vector of $\underset{\sim}{x}$ is $(2, 3)^\top \in \mathbb{R}^2$.

Theorem 0.2 (All bases of a subspace have the same number of vectors)**.** *If* $\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_k$ *and* $\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_l$ *are both bases of a subspace* $\mathcal{S}$ *of* $\mathbb{R}^p$ *(Definition 0.1), then* $k = l$*.*

Proof

Proof. Let $\mathbf{A}$ be the $p \times k$ matrix with columns $\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_k$, and $\mathbf{B}$ the $p \times l$ matrix with columns $\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_l$.

Write A in terms of B. Each $\underset{\sim}{a}_j$ is in $\mathcal{S}$, and the $\underset{\sim}{b}$'s span $\mathcal{S}$ (Definition 5 in Subspaces and Rank), so $\underset{\sim}{a}_j = \mathbf{B}\underset{\sim}{c}_j$ for some $\underset{\sim}{c}_j \in \mathbb{R}^l$ (Definition 8 in Matrix Multiplication). Let $\mathbf{C}$ be the $l \times k$ matrix with columns $\underset{\sim}{c}_1, \dots, \underset{\sim}{c}_k$. Then $\underset{p\times k}{\mathbf{A}} = \underset{p\times l}{\mathbf{B}}\ \underset{l\times k}{\mathbf{C}}$, because column j of $\mathbf{BC}$ is $\mathbf{B}\underset{\sim}{c}_j$.

The columns of C are linearly independent. Suppose $\mathbf{C}\underset{\sim}{x} = \underset{\sim}{0}$ for some $\underset{\sim}{x} \in \mathbb{R}^k$. Then

$$\begin{aligned}
\mathbf{A}\underset{\sim}{x} &= (\mathbf{BC})\,\underset{\sim}{x} && \text{(substitute } \mathbf{A} = \mathbf{BC}\text{)} \\
&= \mathbf{B}\,(\mathbf{C}\underset{\sim}{x}) && \text{(Theorem 1 in Matrix Multiplication)} \\
&= \mathbf{B}\,\underset{\sim}{0} && \text{(substitute } \mathbf{C}\underset{\sim}{x} = \underset{\sim}{0}\text{)} \\
&= \underset{\sim}{0} && \text{(a matrix times the zero vector is } \underset{\sim}{0}\text{)}
\end{aligned}$$

$\mathbf{A}\underset{\sim}{x}$ is the combination $x_1\underset{\sim}{a}_1 + \cdots + x_k\underset{\sim}{a}_k$ (Theorem 3 in Subspaces and Rank), and the $\underset{\sim}{a}$'s are linearly independent (Definition 1 in Subspaces and Rank), so $\underset{\sim}{x} = \underset{\sim}{0}$. So the only combination of the columns of $\mathbf{C}$ that equals $\underset{\sim}{0}$ has all coefficients 0.

Count. $\mathbf{C}$ has k linearly independent columns of length l. More than l vectors in $\mathbb{R}^l$ are never linearly independent (Theorem 5 in Subspaces and Rank), so $k \le l$. Exchanging the roles of the two bases gives $l \le k$, so $k = l$. □

Example 0.5 (Both bases of $\mathbb{R}^2$ in Example 0.1 have two vectors)**.** The bases $(1, 0), (0, 1)$ and $(1, 0), (1, 1)$ of $\mathbb{R}^2$ in Example 0.1 both have 2 vectors, as Theorem 0.2 requires. So no list of 3 vectors is a basis of $\mathbb{R}^2$, which agrees with Example 0.2: the list $(1, 0), (0, 1), (1, 1)$ spans $\mathbb{R}^2$ but is not linearly independent.

Definition 0.4 (Dimension)**.** The **dimension** of a subspace $\mathcal{S}$, written $\dim(\mathcal{S})$, is the number of vectors in any basis of $\mathcal{S}$ (Definition 0.1). Theorem 0.2 makes this number

well defined. By convention, the empty list is the basis of $\{\underset{\sim}{0}\}$, so $\dim(\{\underset{\sim}{0}\}) = 0$.

Example 0.6 (The dimensions of some subspaces)**.**

- $\dim(\{\underset{\sim}{0}\}) = 0$, by the convention in Definition 0.4.
- $\dim(\mathbb{R}^2) = 2$, by Example 0.1. More generally, $\dim(\mathbb{R}^p) = p$: the indicator vectors $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p$ (Definition 21 in Vectors) are linearly independent, since $c_1\underset{\sim}{e}_1 + \cdots + c_p\underset{\sim}{e}_p = (c_1, \ldots, c_p)$, which is $\underset{\sim}{0}_p$ only if every $c_i = 0$; and they span $\mathbb{R}^p$, since any $\underset{\sim}{x} = x_1\underset{\sim}{e}_1 + \cdots + x_p\underset{\sim}{e}_p$.
- The line in Example 3 in Subspaces and Rank has dimension 1: the single vector $(1, 2)$ is linearly independent and spans the line.
- The plane $z = 0$ in Example 6 in Subspaces and Rank has dimension 2, with basis $(1, 0, 0), (0, 1, 0)$.

Exercise 0.5 (Testing whether two vectors can form a basis of $\mathbb{R}^3$)**.** A list of two features cannot describe every point in $\mathbb{R}^3$, and Theorem 0.2 says why in terms of bases. The indicator vectors $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ form a basis of $\mathbb{R}^3$.
Can a list of 2 vectors be a basis of $\mathbb{R}^3$?

Hint

Use Theorem 0.2 with the basis $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$.

Solution

Solution 0.5. No. $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ is a basis of $\mathbb{R}^3$ with 3 vectors, so by Theorem 0.2 every basis of $\mathbb{R}^3$ has 3 vectors, not 2.

Exercise 0.6 (Finding the dimension of a span that includes a multiple)**.** The dimension of a span can be smaller than the number of vectors that span it, for example when one vector is a multiple of another. Let

$$\mathcal{S} = \operatorname{span}\left\{(1, 2, 0)^\top,\ (2, 4, 0)^\top,\ (0, 0, 1)^\top\right\}.$$

Find $\dim(\mathcal{S})$.

Hint

Look for a vector in the list that is a multiple of another, then check that the rest are linearly independent.

Solution

Solution 0.6. $\dim(\mathcal{S}) = 2$. Since $(2,4,0)^\top = 2(1,2,0)^\top$, dropping $(2,4,0)^\top$ leaves the span unchanged, so $(1,2,0)^\top$ and $(0,0,1)^\top$ span $\mathcal{S}$. They are linearly independent: if $c_1(1,2,0)^\top + c_2(0,0,1)^\top = \underset{\sim}{0}$, the first entry gives $c_1 = 0$ and the third gives $c_2 = 0$. So they form a basis of $\mathcal{S}$ with 2 vectors, and by Definition 0.4 the dimension is 2.

Theorem 0.3 (Adding a vector outside the span keeps a list independent)**.** *If $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k$ are linearly independent (Definition 1 in Subspaces and Rank) and $\underset{\sim}{w}$ is not in* $\operatorname{span}\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$ *(Definition 5 in Subspaces and Rank), then $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k, \underset{\sim}{w}$ are linearly independent.*

Proof

Proof. Suppose $c_1\underset{\sim}{v}_1 + \cdots + c_k\underset{\sim}{v}_k + d\,\underset{\sim}{w} = \underset{\sim}{0}$.
The coefficient d is 0. If $d \neq 0$, then

$$\begin{aligned}
\underset{\sim}{w} &= \frac{1}{d}\,(d\,\underset{\sim}{w}) && (d \neq 0)\\
&= \frac{1}{d}\,(-c_1\underset{\sim}{v}_1 - \cdots - c_k\underset{\sim}{v}_k) && \text{(solve the supposed equation for } d\,\underset{\sim}{w})\\
&= \left(-\frac{c_1}{d}\right)\underset{\sim}{v}_1 + \cdots + \left(-\frac{c_k}{d}\right)\underset{\sim}{v}_k, && \text{(distribute } \tfrac{1}{d})
\end{aligned}$$

which puts $\underset{\sim}{w}$ in $\operatorname{span}\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$, contrary to the assumption.
The other coefficients are 0. With $d = 0$, the equation reads $c_1\underset{\sim}{v}_1 + \cdots + c_k\underset{\sim}{v}_k = \underset{\sim}{0}$, so every $c_i = 0$, because $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k$ are linearly independent. □

Example 0.7 (Adding $(0,0,1)$ to two independent vectors in $\mathbb{R}^3$)**.** The vectors $(1,0,0)$ and $(0,1,0)$ are linearly independent, and their span is the plane $z = 0$ (Example 6 in Subspaces and Rank). The vector $(0,0,1)$ has third entry 1, so it is not in that plane, and by Theorem 0.3 the three vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are linearly independent. Adding $(1,1,0)$ instead would not work: $(1,1,0)$ is in the span, and $(1,0,0) + (0,1,0) - (1,1,0) = \underset{\sim}{0}$.

Exercise 0.7 (Adding a third feature to an independent pair)**.** Adding a new feature keeps a list of features linearly independent when the new one is not a combination of the old ones, which is what Theorem 0.3 says. Let

$$\underset{\sim}{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \underset{\sim}{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \underset{\sim}{w} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Are $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ and $\underset{\sim}{w}$ linearly independent?

Hint

Check whether $\underset{\sim}{w}$ is in $\operatorname{span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$, then use Theorem 0.3.

Solution

Solution 0.7. Yes. $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are linearly independent: $c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 = (c_1, c_2, 0)^\top$ is $\underset{\sim}{0}$ only for $c_1 = c_2 = 0$. Every vector in their span has third entry 0, and $\underset{\sim}{w}$ has third entry 1, so $\underset{\sim}{w}$ is not in the span. By Theorem 0.3, $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ and $\underset{\sim}{w}$ are linearly independent.

Theorem 0.4 (An independent list extends to a basis)**.** *If $\mathcal{S}$ is a subspace of $\mathbb{R}^p$ (Definition 4 in Subspaces and Rank) and $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k \in \mathcal{S}$ are linearly independent (Definition 1 in Subspaces and Rank), then adding vectors of $\mathcal{S}$ to the list gives a basis of $\mathcal{S}$ (Definition 0.1). Starting from the empty list shows that every subspace has a basis.*

Proof

Proof. Repeat the following step. If the current list spans $\mathcal{S}$, stop: it is linearly independent and spans $\mathcal{S}$, so it is a basis. Otherwise, some $\underset{\sim}{w} \in \mathcal{S}$ is not in the span of the current list; add $\underset{\sim}{w}$ to the list, which stays linearly independent by Theorem 0.3.
Every list produced is a linearly independent list of vectors in $\mathbb{R}^p$, so it has at most p vectors (Theorem 5 in Subspaces and Rank). Each step adds one vector, so the process stops after at most $p - k$ steps, and when it stops the list is a basis of $\mathcal{S}$.
For the empty list, which is linearly independent and spans $\{\underset{\sim}{0}\}$ by the conventions in Definition 5 in Subspaces and Rank, the first step asks whether $\mathcal{S} = \{\underset{\sim}{0}\}$: if so, the empty list is already a basis; if not, the process adds vectors as described. □

Example 0.8 (Extending $(1, 1, 0)$ to a basis of $\mathbb{R}^3$)**.** Start with the single vector $(1, 1, 0)$.

1. $(1, 0, 0)$ is not in $\operatorname{span}\{(1, 1, 0)\}$, because every multiple $c\,(1, 1, 0)$ has equal first and second entries. Add it.
2. $(0, 0, 1)$ is not in $\operatorname{span}\{(1, 1, 0), (1, 0, 0)\}$, because every combination of those two vectors has third entry 0. Add it.
3. The list $(1, 1, 0), (1, 0, 0), (0, 0, 1)$ spans $\mathbb{R}^3$: $(x, y, z) = y\,(1, 1, 0) + (x - y)\,(1, 0, 0) + z\,(0, 0, 1)$. Stop.

The result is a basis of $\mathbb{R}^3$ that contains the starting vector.

Exercise 0.8 (Extending one vector to a basis of $\mathbb{R}^3$)**.** Theorem 0.4 says that any linearly independent list can be completed to a basis. Let

$$\underset{\sim}{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Find two indicator vectors that extend $\underset{\sim}{v}_1$ to a basis of $\mathbb{R}^3$.

Hint

Add indicator vectors one at a time, each outside the span of the list so far, as Theorem 0.3 allows.

Solution

Solution 0.8. One answer is $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$. The list $\underset{\sim}{v}_1$ is linearly independent, since $\underset{\sim}{v}_1 \neq \underset{\sim}{0}$. $\underset{\sim}{e}_2 = (0,1,0)^\top$ is not a multiple of $\underset{\sim}{v}_1$, whose first two entries match, so by Theorem 0.3 the list $\underset{\sim}{v}_1, \underset{\sim}{e}_2$ is linearly independent. Every vector in its span has third entry 0, so $\underset{\sim}{e}_3$ is outside it, and $\underset{\sim}{v}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ is linearly independent too. These three vectors span $\mathbb{R}^3$, because $(x,y,z)^\top = x\underset{\sim}{v}_1 + (y-x)\underset{\sim}{e}_2 + z\underset{\sim}{e}_3$, so they form a basis of $\mathbb{R}^3$. The pair $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_3$ works too.

Theorem 0.5 (An independent list in a subspace has at most dim vectors)**.** *Let $\mathcal{S}$ be a subspace of $\mathbb{R}^p$ with $\dim(\mathcal{S}) = d$ (Definition 0.4).*

1. *Any linearly independent list of vectors in $\mathcal{S}$ has at most d vectors.*
2. *If $\mathcal{T}$ is a subspace with $\mathcal{T} \subseteq \mathcal{S}$, then $\dim(\mathcal{T}) \leq \dim(\mathcal{S})$.*

Proof

Proof. **Part 1.** If $d = 0$, then $\mathcal{S} = \{\underset{\sim}{0}\}$, so every vector in a nonempty list from $\mathcal{S}$ is $\underset{\sim}{0}$, and $1 \cdot \underset{\sim}{0} = \underset{\sim}{0}$ shows the list is not linearly independent; only the empty list, with $0 = d$ vectors, is. Now let $d \geq 1$, and let $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_d$ be a basis of $\mathcal{S}$, and let $\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_k \in \mathcal{S}$ be linearly independent. Each $\underset{\sim}{u}_j$ is in $\mathcal{S}$, so $\underset{\sim}{u}_j = c_{1j}\underset{\sim}{a}_1 + \cdots + c_{dj}\underset{\sim}{a}_d$ for some coordinate vector $\underset{\sim}{c}_j = (c_{1j}, \ldots, c_{dj}) \in \mathbb{R}^d$. Suppose $k > d$. Then $\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_k$ are $k > d$ vectors in $\mathbb{R}^d$, so some $x_1, \ldots, x_k$, not all zero, satisfy $x_1\underset{\sim}{c}_1 + \cdots + x_k\underset{\sim}{c}_k = \underset{\sim}{0}_d$ (Theorem 5 in Subspaces and Rank); entry i of that equation says $\sum_{j=1}^{k} x_j c_{ij} = 0$. Then

$$
\begin{aligned}
\sum_{j=1}^{k} x_j \underset{\sim}{u}_j &= \sum_{j=1}^{k} x_j \sum_{i=1}^{d} c_{ij} \underset{\sim}{a}_i && \text{(substitute each } \underset{\sim}{u}_j\text{)} \\
&= \sum_{j=1}^{k} \sum_{i=1}^{d} x_j c_{ij}\, \underset{\sim}{a}_i && \text{(distribute each } x_j \text{ over the inner sum)} \\
&= \sum_{i=1}^{d} \sum_{j=1}^{k} x_j c_{ij}\, \underset{\sim}{a}_i && \text{(swap the order of the finite sums)} \\
&= \sum_{i=1}^{d} \left(\sum_{j=1}^{k} x_j c_{ij} \right) \underset{\sim}{a}_i && \text{(factor } \underset{\sim}{a}_i \text{ out of the inner sum)} \\
&= \sum_{i=1}^{d} 0 \cdot \underset{\sim}{a}_i && \text{(entry } i \text{ of } \textstyle\sum_j x_j \underset{\sim}{c}_j = \underset{\sim}{0}_d\text{)} \\
&= \underset{\sim}{0}_p, && \text{(a zero multiple of a vector is } \underset{\sim}{0}\text{)}
\end{aligned}
$$

a combination of $\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_k$ with coefficients not all zero that equals $\underset{\sim}{0}_p$. That contradicts their linear independence, so $k \leq d$.

Part 2. $\mathcal{T}$ has a basis (Theorem 0.4) with $\dim(\mathcal{T})$ vectors. Those vectors are linearly independent and lie in $\mathcal{S}$, so by Part 1 there are at most $\dim(\mathcal{S})$ of them. □

Example 0.9 (Subspaces of a plane have dimension at most two)**.** The plane $\mathcal{S} = \{(x, y, 0) : x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$ has dimension 2 (Example 0.6). By Theorem 0.5, no three vectors with third entry 0 are linearly independent; for instance, $(1, 0, 0) + (0, 1, 0) - (1, 1, 0) = \underset{\sim}{0}$. The line $\mathcal{T} = \text{span}\{(1, 1, 0)\}$ lies in $\mathcal{S}$, and

$$
\begin{aligned}
\dim(\mathcal{T}) &= 1 \\
&\leq 2 \\
&= \dim(\mathcal{S}).
\end{aligned}
$$

Exercise 0.9 (Testing whether four independent vectors fit in a three-dimensional subspace)**.** Theorem 0.5 caps how many linearly independent vectors a subspace can hold. Let $\mathcal{S}$ be a subspace of $\mathbb{R}^5$ with dimension 3.
Can $\mathcal{S}$ contain 4 linearly independent vectors?

Hint

Use part 1 of Theorem 0.5 with $d = 3$.

Solution

Solution 0.9. No. By part 1 of Theorem 0.5, a linearly independent list of vectors in $\mathcal{S}$ has at most $\dim(\mathcal{S}) = 3$ vectors, so it cannot have 4.

Theorem 0.6 (A subspace inside another of the same dimension equals it)**.** *If $\mathcal{T}$ and $\mathcal{S}$ are subspaces of $\mathbb{R}^p$ with $\mathcal{T} \subseteq \mathcal{S}$ and $\dim(\mathcal{T}) = \dim(\mathcal{S})$ (Definition 0.4), then $\mathcal{T} = \mathcal{S}$.*

Proof

Proof. Let $d = \dim(\mathcal{S})$, and let $\underset{\sim}{t}_1, \dots, \underset{\sim}{t}_d$ be a basis of $\mathcal{T}$ (Theorem 0.4); it has d vectors because $\dim(\mathcal{T}) = d$. (When $d = 0$ it is the empty list, whose span is $\{\underset{\sim}{0}\}$ by the convention in Definition 5 in Subspaces and Rank, and the argument is unchanged.) Suppose some $\underset{\sim}{s} \in \mathcal{S}$ is not in $\operatorname{span}\{\underset{\sim}{t}_1, \dots, \underset{\sim}{t}_d\} = \mathcal{T}$. Then $\underset{\sim}{t}_1, \dots, \underset{\sim}{t}_d, \underset{\sim}{s}$ are linearly independent (Theorem 0.3), and they are $d + 1$ vectors of $\mathcal{S}$, which contradicts part 1 of Theorem 0.5. So every vector of $\mathcal{S}$ is in $\mathcal{T}$, that is, $\mathcal{S} \subseteq \mathcal{T}$. Together with $\mathcal{T} \subseteq \mathcal{S}$, this inclusion gives $\mathcal{T} = \mathcal{S}$. □

Example 0.10 (Two independent vectors in a plane span it)**.** Let $\mathcal{S} = \{(x, y, 0) : x, y \in \mathbb{R}\}$, the plane $z = 0$, which has dimension 2 (Example 0.6), and let $\mathcal{T} = \operatorname{span}\{(1, 0, 0), (1, 1, 0)\}$. Both vectors have third entry 0, so $\mathcal{T} \subseteq \mathcal{S}$. They are linearly independent: $c_1(1, 0, 0) + c_2(1, 1, 0) = (c_1 + c_2, c_2, 0)$, which is $\underset{\sim}{0}$ only if $c_2 = 0$ and then $c_1 = 0$. So $\dim(\mathcal{T}) = 2$, and Theorem 0.6 gives $\mathcal{T} = \mathcal{S}$ without solving for any coefficients. As a check, $(x, y, 0) = (x - y)\,(1, 0, 0) + y\,(1, 1, 0)$.

Example 0.11 (Equal dimensions without containment)**.** The lines $\operatorname{span}\{(1, 0, 0)\}$ and $\operatorname{span}\{(0, 1, 0)\}$ in $\mathbb{R}^3$ both have dimension 1, but they are not equal: $(1, 0, 0)$ is in the first and not the second. Theorem 0.6 does not apply, because neither line contains the other.

Exercise 0.10 (Showing that a span of two vectors is all of $\mathbb{R}^2$)**.** Theorem 0.6 shows that two subspaces coincide by comparing dimensions instead of checking every vector. Let

$$\mathcal{T} = \operatorname{span}\Big\{(1, 2)^\top,\ (3, 1)^\top\Big\}.$$

Is $\mathcal{T}$ all of $\mathbb{R}^2$?

Hint

Find $\dim(\mathcal{T})$ and $\dim(\mathbb{R}^2)$, then use Theorem 0.6 with $\mathcal{S} = \mathbb{R}^2$.

Solution

Solution 0.10. Yes. The two vectors are linearly independent: if $c_1(1,2)^\top + c_2(3,1)^\top = \underset{\sim}{0}$, then $c_1 + 3c_2 = 0$ and $2c_1 + c_2 = 0$. Subtracting twice the first equation from the second gives $-5c_2 = 0$, so $c_2 = 0$ and then $c_1 = 0$. So they form a basis of $\mathcal{T}$, and $\dim(\mathcal{T}) = 2$. The indicator vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ form a basis of $\mathbb{R}^2$, so $\dim(\mathbb{R}^2) = 2$. Since $\mathcal{T} \subseteq \mathbb{R}^2$ and the dimensions agree, Theorem 0.6 gives $\mathcal{T} = \mathbb{R}^2$.

[1] Wikipedia contributors, "Basis (linear algebra) — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Basis_(linear_algebra)

[2] Wikipedia contributors, "Coordinate vector — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Coordinate_vector