Affine Subspaces and Half-Spaces

Last modified: 2026-11-24 09:19:45 (PST)

Affine subspaces and half-spaces

Definition 0.1 (Affine subspace)**.** An **affine subspace** [1] of $\mathbb{R}^p$ is a set of the form

$$\{\underset{\sim}{x}_0 + \underset{\sim}{s} : \underset{\sim}{s} \in \mathcal{S}\}$$

for some vector $\underset{\sim}{x}_0 \in \mathbb{R}^p$ and some subspace $\mathcal{S}$ of $\mathbb{R}^p$ (Definition 4 in Subspaces and Rank): the image of $\mathcal{S}$ under the translation by $\underset{\sim}{x}_0$ (Definition 6 in Linear and Affine Maps).

Example 0.1 (A line that misses the origin is an affine subspace)**.** The line $\mathcal{T} = \{(x, 1) : x \in \mathbb{R}\}$ of Example 4 in Subspaces and Rank is not a subspace, but it is an affine subspace, with $\underset{\sim}{x}_0 = (0, 1)$ and $\mathcal{S} = \text{span}\{(1, 0)\}$ (Definition 5 in Subspaces and Rank):

$$\begin{aligned} (0,1) + c\,(1,0) &= (0 + c,\ 1 + 0) && \text{(Definition 9 in Matrices, Definition 9 in Vectors)} \\ &= (c, 1), && \text{(add)} \end{aligned}$$

and as c ranges over $\mathbb{R}$, $(c, 1)$ ranges over all of $\mathcal{T}$. Every subspace $\mathcal{S}$ is also an affine subspace, with $\underset{\sim}{x}_0 = \underset{\sim}{0}$.

Exercise 0.1 (Writing a line as an affine subspace)**.** A line such as $y = 2x + 3$ in $\mathbb{R}^2$ misses the origin, so it is not a subspace, but it is an affine subspace. Let $\mathcal{T} = \{(x, 2x + 3) : x \in \mathbb{R}\}$.
Find a vector $\underset{\sim}{x}_0$ and a subspace $\mathcal{S}$ with $\mathcal{T} = \{\underset{\sim}{x}_0 + \underset{\sim}{s} : \underset{\sim}{s} \in \mathcal{S}\}$.

> **Hint**
>
> Take $\underset{\sim}{x}_0$ to be the point of $\mathcal{T}$ with $x = 0$, and see what is left of $(x, 2x + 3)$ once you subtract it.

Solution

Solution 0.1. Take $\underset{\sim}{x}_0 = (0,3)$, and let $\mathcal{S} = \text{span}\{(1,2)\}$ be the span of $(1,2)$. For any $c \in \mathbb{R}$,

$$(0,3) + c\,(1,2) = (c, 2c+3),$$

so every $c \in \mathbb{R}$ gives a point $(c, 2c+3)$ of $\mathcal{T}$, and every point of $\mathcal{T}$ arises this way.

Theorem 0.1 (A hyperplane is a translated subspace of dimension $p-1$)**.** *Let $\underset{\sim}{w} \in \mathbb{R}^p$ be nonzero, let $b \in \mathbb{R}$, let $\mathcal{H} = \{\underset{\sim}{x} \in \mathbb{R}^p : \underset{\sim}{w}^\top \underset{\sim}{x} + b = 0\}$ be the hyperplane with normal vector $\underset{\sim}{w}$ and offset b (Definition 10 in Linear and Affine Maps), and let $\mathcal{H}_0 = \{\underset{\sim}{x} \in \mathbb{R}^p : \underset{\sim}{w}^\top \underset{\sim}{x} = 0\}$ be the hyperplane with the same normal vector and offset 0.*

1. *$\mathcal{H}_0$ is a subspace of $\mathbb{R}^p$ (Definition 4 in Subspaces and Rank) of dimension $p-1$ (Definition 4 in Bases and Dimension).*
2. *$\mathcal{H}$ is not empty, and for any $\underset{\sim}{x}_0 \in \mathcal{H}$, $\mathcal{H} = \{\underset{\sim}{x}_0 + \underset{\sim}{s} : \underset{\sim}{s} \in \mathcal{H}_0\}$, so $\mathcal{H}$ is an affine subspace (Definition 0.1).*
3. *For any $\underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{H}$, the normal vector is orthogonal to their difference: $\underset{\sim}{w} \perp (\underset{\sim}{x} - \underset{\sim}{y})$ (Definition 22 in Vectors).*

Proof

Proof. Here $\underset{\sim}{w}^\top$ is a $1 \times p$ matrix, and $\underset{\sim}{w}^\top \underset{\sim}{x} = \underset{\sim}{w} \cdot \underset{\sim}{x}$ (Example 3 in Matrix Multiplication).
Part 1. $\mathcal{H}_0$ is the null space $\mathcal{N}(\underset{\sim}{w}^\top)$ (Definition 3 in Column Space and Null Space), so it is a subspace (Theorem 2 in Column Space and Null Space). The p columns of $\underset{\sim}{w}^\top$ are the numbers $w_1, \ldots, w_p$. Some $w_j \neq 0$, because $\underset{\sim}{w} \neq 0$, and that column on its own is linearly independent ($c\,w_j = 0$ forces $c = 0$), so $\text{rank}(\underset{\sim}{w}^\top) \geq 1$ (Definition 2 in Subspaces and Rank); and $\text{rank}(\underset{\sim}{w}^\top) \leq \min\{1, p\} = 1$ (Corollary 1 in Rank-Nullity and Rank Factorization). So

$$\begin{aligned}
\dim(\mathcal{H}_0) &= \text{nullity}(\underset{\sim}{w}^\top) && \text{(Definition 1 in Rank-Nullity and Rank Factorization)} \\
&= p - \text{rank}(\underset{\sim}{w}^\top) && \text{(Theorem 2 in Rank-Nullity and Rank Factorization, for the } p \text{ columns of } \underset{\sim}{w}^\top) \\
&= p - 1. && \text{(the rank is 1)}
\end{aligned}$$

Part 2. $\underset{\sim}{w} \cdot \underset{\sim}{w} = w_1^2 + \cdots + w_p^2 > 0$, because some $w_j \neq 0$. So $\underset{\sim}{x}^* \overset{\text{def}}{=} -\frac{b}{\underset{\sim}{w} \cdot \underset{\sim}{w}}\,\underset{\sim}{w}$ is defined, and

$$\begin{aligned}
\underset{\sim}{w}^\top \underset{\sim}{x}^* + b &= -\frac{b}{\underset{\sim}{w} \cdot \underset{\sim}{w}}\,(\underset{\sim}{w}^\top \underset{\sim}{w}) + b && \text{(homogeneity of } \underset{\sim}{x} \mapsto \underset{\sim}{w}^\top \underset{\sim}{x}\text{, Theorem 2 in Linear and Affine Maps)} \\
&= -b + b && (\underset{\sim}{w}^\top \underset{\sim}{w} = \underset{\sim}{w} \cdot \underset{\sim}{w}) \\
&= 0, && \text{(arithmetic)}
\end{aligned}$$

so $\underset{\sim}{x}^* \in \mathcal{H}$, and $\mathcal{H}$ is not empty. Now take any $\underset{\sim}{x}_0 \in \mathcal{H}$, so $\underset{\sim}{w}^\top \underset{\sim}{x}_0 = -b$. For any $\underset{\sim}{x} \in \mathbb{R}^p$,

$$\begin{aligned} \underset{\sim}{w}^\top(\underset{\sim}{x} - \underset{\sim}{x}_0) &= \underset{\sim}{w}^\top \underset{\sim}{x} - \underset{\sim}{w}^\top \underset{\sim}{x}_0 && \text{(Theorem 4 in Subspaces and Rank, with coefficients 1 and } -1) \\ &= \underset{\sim}{w}^\top \underset{\sim}{x} - (-b) && (\underset{\sim}{x}_0 \in \mathcal{H}) \\ &= \underset{\sim}{w}^\top \underset{\sim}{x} + b, && \text{(arithmetic)} \end{aligned}$$

so $\underset{\sim}{x} \in \mathcal{H}$ exactly when $\underset{\sim}{x} - \underset{\sim}{x}_0 \in \mathcal{H}_0$, that is, exactly when $\underset{\sim}{x} = \underset{\sim}{x}_0 + \underset{\sim}{s}$ with $\underset{\sim}{s} = \underset{\sim}{x} - \underset{\sim}{x}_0 \in \mathcal{H}_0$. $\mathcal{H}_0$ is a subspace (part 1), so $\mathcal{H}$ has the form in Definition 0.1.

Part 3. For $\underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{H}$, both $\underset{\sim}{w}^\top \underset{\sim}{x}$ and $\underset{\sim}{w}^\top \underset{\sim}{y}$ equal $-b$, so

$$\begin{aligned} \underset{\sim}{w} \cdot (\underset{\sim}{x} - \underset{\sim}{y}) &= \underset{\sim}{w}^\top \underset{\sim}{x} - \underset{\sim}{w}^\top \underset{\sim}{y} && \text{(Theorem 4 in Subspaces and Rank, with coefficients 1 and } -1) \\ &= (-b) - (-b) && (\underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{H}) \\ &= 0. && \text{(arithmetic)} \end{aligned}$$

□

Example 0.2 (The plane $x_1 + x_2 + x_3 = 1$ in $\mathbb{R}^3$)**.** Take $\underset{\sim}{w} = (1,1,1)$ and $b = -1$, so $\mathcal{H} = \{\underset{\sim}{x} : x_1 + x_2 + x_3 - 1 = 0\}$, the plane that Example 5 in Subspaces and Rank showed is not a subspace. By part 1, $\mathcal{H}_0 = \{\underset{\sim}{x} : x_1 + x_2 + x_3 = 0\}$ is a subspace of dimension $3 - 1 = 2$.

- $\underset{\sim}{x}_0 = (1,0,0)$ is in $\mathcal{H}$, since $1 + 0 + 0 - 1 = 0$, and so is $\underset{\sim}{y} = (0,1,0)$.
- Their difference $\underset{\sim}{y} - \underset{\sim}{x}_0 = (-1,1,0)$ has entries summing to $-1 + 1 + 0 = 0$, so it is in $\mathcal{H}_0$, and $\underset{\sim}{y} = \underset{\sim}{x}_0 + (-1,1,0)$, as part 2 says.
- $$\begin{aligned} \underset{\sim}{w} \cdot (-1,1,0) &= -1 + 1 + 0 \\ &= 0, \end{aligned}$$

  as part 3 says.
- The point built in the proof is

  $$\begin{aligned} \underset{\sim}{x}^* &= -\frac{-1}{3}(1,1,1) \\ &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), \end{aligned}$$

  and $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} - 1 = 0$.

Exercise 0.2 (Checking that a normal vector is orthogonal to a hyperplane)**.** Part 3 of Theorem 0.1 says that, for a hyperplane, its normal vector is orthogonal to the difference of any two of its points. Let $\mathcal{H} = \{\underset{\sim}{x} \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 - 4 = 0\}$, with normal vector $\underset{\sim}{w} = (2, -1, 1)$. The points $\underset{\sim}{x} = (2, 0, 0)$ and $\underset{\sim}{y} = (0, 0, 4)$ are both in $\mathcal{H}$.
Check that $\underset{\sim}{w} \cdot (\underset{\sim}{x} - \underset{\sim}{y}) = 0$.

Hint

Compute $\underset{\sim}{x} - \underset{\sim}{y}$ first.

Solution

Solution 0.2. Here $\underset{\sim}{x} - \underset{\sim}{y} = (2, 0, -4)$, and

$$\underset{\sim}{w} \cdot (\underset{\sim}{x} - \underset{\sim}{y}) = 2 \cdot 2 + (-1) \cdot 0 + 1 \cdot (-4) = 0,$$

as part 3 of Theorem 0.1 says.

Theorem 0.2 (Where a hyperplane passes, and its shape in $\mathbb{R}^2$ and $\mathbb{R}^3$)**.** *Let* $\mathcal{H} = \{\underset{\sim}{x} \in \mathbb{R}^p : \underset{\sim}{w} \cdot \underset{\sim}{x} + b = 0\}$ *be a hyperplane (Definition 10 in Linear and Affine Maps).*

1. $\mathcal{H}$ *contains the origin* $\underset{\sim}{0}$ *exactly when* $b = 0$.
2. *If* $p = 2$*, then* $\mathcal{H}$ *is a line:* $\mathcal{H} = \{\underset{\sim}{x}_0 + t\,\underset{\sim}{v} : t \in \mathbb{R}\}$ *for some* $\underset{\sim}{x}_0 \in \mathbb{R}^2$ *and some nonzero* $\underset{\sim}{v} \in \mathbb{R}^2$.
3. *If* $p = 3$*, then* $\mathcal{H}$ *is a plane:* $\mathcal{H} = \{\underset{\sim}{x}_0 + s\,\underset{\sim}{v}_1 + t\,\underset{\sim}{v}_2 : s, t \in \mathbb{R}\}$ *for some* $\underset{\sim}{x}_0 \in \mathbb{R}^3$ *and some linearly independent* $\underset{\sim}{v}_1, \underset{\sim}{v}_2 \in \mathbb{R}^3$.

Proof

Proof. **Part 1.** Since $\underset{\sim}{w} \cdot \underset{\sim}{0} = 0$,

$$\begin{aligned} \underset{\sim}{w} \cdot \underset{\sim}{0} + b &= 0 + b && \text{(the dot product with } \underset{\sim}{0} \text{ is 0)} \\ &= b, && \text{(Theorem 1 in Sums, Products and Quotients)} \end{aligned}$$

so $\underset{\sim}{0} \in \mathcal{H}$ exactly when $b = 0$.
Parts 2 and 3. By Theorem 0.1, $\mathcal{H} = \{\underset{\sim}{x}_0 + \underset{\sim}{s} : \underset{\sim}{s} \in \mathcal{H}_0\}$ for any $\underset{\sim}{x}_0 \in \mathcal{H}$, where $\mathcal{H}_0$ is a subspace of $\mathbb{R}^p$ of dimension $p - 1$.
For $p = 2$, $\mathcal{H}_0$ has dimension 1, so it has a basis (Definition 1 in Bases and Dimension) with one vector $\underset{\sim}{v}$. That vector is nonzero, and $\mathcal{H}_0 = \{t\,\underset{\sim}{v} : t \in \mathbb{R}\}$.
For $p = 3$, $\mathcal{H}_0$ has dimension 2, so it has a basis with two vectors $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$. They are linearly independent, and $\mathcal{H}_0 = \{s\,\underset{\sim}{v}_1 + t\,\underset{\sim}{v}_2 : s, t \in \mathbb{R}\}$. □

Example 0.3 (A line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$, checked on a grid of points)**.** In $\mathbb{R}^2$, take $\underset{\sim}{w} = (1, 1)$ and $b = -1$, the line of Example 11 in Linear and Affine Maps. The point $\underset{\sim}{x}_0 = -\frac{b}{\underset{\sim}{w}\cdot\underset{\sim}{w}}\,\underset{\sim}{w}$ is on it, and the vector $\underset{\sim}{v} = (-w_2, w_1)$ is nonzero with $\underset{\sim}{w}\cdot\underset{\sim}{v} = 0$, so it spans the subspace $\mathcal{H}_0$ of Theorem 0.1. In $\mathbb{R}^3$, take $\underset{\sim}{w} = (1, 1, 1)$ and $b = -1$, the plane of Example 0.2, with the independent vectors $\underset{\sim}{v}_1 = (1, -1, 0)$ and $\underset{\sim}{v}_2 = (0, 1, -1)$ in $\mathcal{H}_0$. The code evaluates $\underset{\sim}{w}\cdot\underset{\sim}{x} + b$ at the origin and at points $\underset{\sim}{x}_0 + t\,\underset{\sim}{v}$ and $\underset{\sim}{x}_0 + s\,\underset{\sim}{v}_1 + t\,\underset{\sim}{v}_2$.

```
level <- function(w, b, x) sum(w * x) + b

w2 <- c(1, 1)
b2 <- -1
point_line <- -b2 / sum(w2 * w2) * w2
v <- c(-w2[2], w2[1])
line_levels <- sapply(-2:2, function(t) level(w2, b2, point_line + t * v))

w3 <- c(1, 1, 1)
b3 <- -1
point_plane <- -b3 / sum(w3 * w3) * w3
v1 <- c(1, -1, 0)
v2 <- c(0, 1, -1)
grid <- expand.grid(s = -2:2, t = -2:2)
plane_levels <- mapply(
  function(s, t) level(w3, b3, point_plane + s * v1 + t * v2),
  grid$s, grid$t
)

c(
  origin_in_R2 = level(w2, b2, c(0, 0)),
  origin_in_R3 = level(w3, b3, c(0, 0, 0)),
  origin_with_b0 = level(w3, 0, c(0, 0, 0))
)
#>   origin_in_R2   origin_in_R3 origin_with_b0
#>             -1             -1              0
```

At the origin the left side equals the offset, so the origin is on the line only if b is 0: it gives -1 for the line and -1 for the plane, and 0 once the offset is changed to 0. All 5 points $\underset{\sim}{x}_0 + t\,\underset{\sim}{v}$ on the line and all 25 points $\underset{\sim}{x}_0 + s\,\underset{\sim}{v}_1 + t\,\underset{\sim}{v}_2$ on the plane satisfy the equation up to rounding error: 30 of the 30 points have $|\underset{\sim}{w}\cdot\underset{\sim}{x} + b| < 10^{-10}$.

Exercise 0.3 (Describing a line in $\mathbb{R}^2$ by a point and a vector)**.** Part 2 of Theorem 0.2 says that a hyperplane in $\mathbb{R}^2$ is a line $\{\underset{\sim}{x}_0 + t\,\underset{\sim}{v} : t \in \mathbb{R}\}$ for a point $\underset{\sim}{x}_0$ and a nonzero vector $\underset{\sim}{v}$. Let $\mathcal{H} = \{\underset{\sim}{x} \in \mathbb{R}^2 : 2x_1 + x_2 - 4 = 0\}$.
Find a point $\underset{\sim}{x}_0$ and a nonzero vector $\underset{\sim}{v}$ with $\mathcal{H} = \{\underset{\sim}{x}_0 + t\,\underset{\sim}{v} : t \in \mathbb{R}\}$.

> **Hint**
>
> Find one point of $\mathcal{H}$ by setting $x_2 = 0$, and take $\underset{\sim}{v} = (-w_2, w_1)$ as in Example 0.3.

Solution

Solution 0.3. Setting $x_2 = 0$ gives $x_1 = 2$, so $\underset{\sim}{x}_0 = (2, 0)$ is in $\mathcal{H}$. With $\underset{\sim}{w} = (2, 1)$, take $\underset{\sim}{v} = (-1, 2)$. Then $\underset{\sim}{x}_0 + t\,\underset{\sim}{v} = (2 - t, 2t)$, and

$$2\,(2 - t) + 2t - 4 = 0$$

for every t. Conversely, a point (x_1, x_2) of $\mathcal{H}$ has $x_2 = 4 - 2x_1$, so it is $\underset{\sim}{x}_0 + t\,\underset{\sim}{v}$ with $t = 2 - x_1$.

Theorem 0.3 (The graph of an affine function is a hyperplane)**.** *Let $\underset{\sim}{w} \in \mathbb{R}^p$, $b \in \mathbb{R}$. The graph of the function $\underset{\sim}{x} \mapsto \underset{\sim}{w} \cdot \underset{\sim}{x} + b$ is the set*

$$\left\{ \begin{bmatrix} \underset{\sim}{x} \\ y \end{bmatrix} \in \mathbb{R}^{p+1} : y = \underset{\sim}{w} \cdot \underset{\sim}{x} + b \right\}.$$

This set is a hyperplane in $\mathbb{R}^{p+1}$, with normal vector $\begin{bmatrix} \underset{\sim}{w} \\ -1 \end{bmatrix}$.

Proof

Proof. For $\begin{bmatrix} \underset{\sim}{x} \\ y \end{bmatrix} \in \mathbb{R}^{p+1}$,

$$\begin{aligned} y = \underset{\sim}{w} \cdot \underset{\sim}{x} + b &\iff \underset{\sim}{w} \cdot \underset{\sim}{x} - y + b = 0 && \text{(subtract } y \text{ from both sides)} \\ &\iff \begin{bmatrix} \underset{\sim}{w} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{x} \\ y \end{bmatrix} + b = 0. && \text{(dot product of stacked vectors)} \end{aligned}$$

So the set is the solution set of the equation of a hyperplane, with normal vector $\begin{bmatrix} \underset{\sim}{w} \\ -1 \end{bmatrix}$. That vector is nonzero, because its last entry is -1. □

Example 0.4 (The graph of a line is a line in the plane)**.** Take $p = 1$, $\underset{\sim}{w} = 2$, $b = 1$. The graph of $y = 2x + 1$ is the set of points (x, y) in $\mathbb{R}^2$ with $2x - y + 1 = 0$. By Theorem 0.3, this is a hyperplane with normal vector $(2, -1)$. The point $(0, 1)$ is on it, since $2 \cdot 0 - 1 + 1 = 0$. The point $(0, 0)$ is off it, since $2 \cdot 0 - 0 + 1 = 1 \neq 0$. Figure 1 shows the line, the two points, and the normal vector drawn from $(0, 1)$.

```
arrow_length <- grid::unit(0.15, "inches")
normal_arrow <- grid::arrow(length = arrow_length)
point_x <- c(0, 0)
point_y <- c(1, 0)
point_label <- c("on the line", "off the line")
points <- tibble::tibble(x = point_x, y = point_y, label = point_label)
normal <- tibble::tibble(x = 0, y = 1, xend = 2, yend = 0)
normal_aes <- ggplot2::aes(x = x, y = y, xend = xend, yend = yend)
x_limits <- c(-2, 3)
y_limits <- c(-2, 4)
ggplot2::ggplot(points) +
  ggplot2::aes(x = x, y = y, colour = label) +
  ggplot2::geom_abline(intercept = 1, slope = 2) +
  ggplot2::geom_segment(
    mapping = normal_aes,
    data = normal,
    arrow = normal_arrow,
    inherit.aes = FALSE
  ) +
  ggplot2::geom_point(size = 3) +
  ggplot2::coord_fixed(xlim = x_limits, ylim = y_limits) +
  ggplot2::labs(x = "x", y = "y", colour = NULL)
```

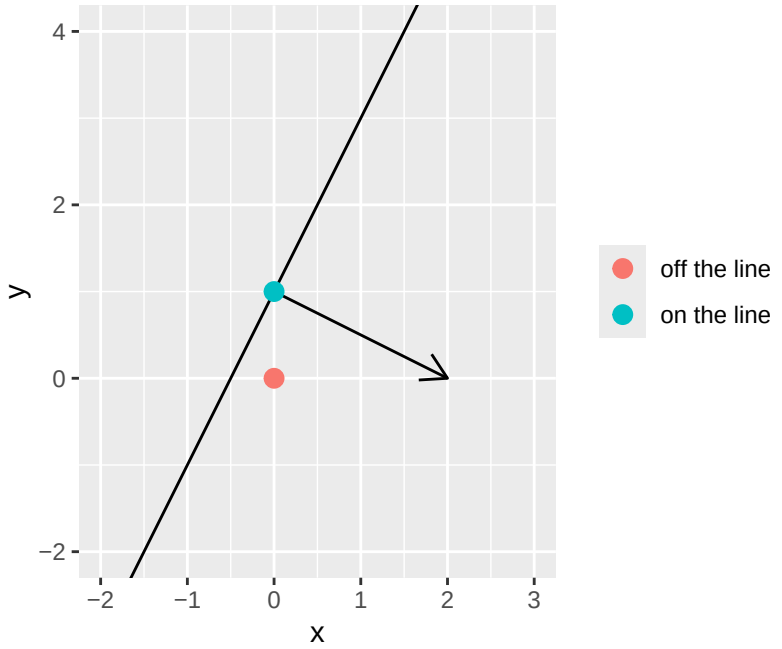

Figure 1: The graph of $y = 2x + 1$, a line in the plane, with its normal vector $(2, -1)$ drawn from the point $(0, 1)$ on it and the point $(0, 0)$ off it

Exercise 0.4 (Finding the normal vector of the graph of a regression surface)**.** Theorem 0.3 says that the graph of $\underset{\sim}{x} \mapsto \underset{\sim}{w} \cdot \underset{\sim}{x} + b$ is a hyperplane in $\mathbb{R}^{p+1}$. Let $f(x_1, x_2) = 3x_1 - x_2 + 2$, a regression surface with two features.
Find a normal vector for the graph of f in $\mathbb{R}^3$.

Hint

Read off $\underset{\sim}{w}$ and use the normal vector $\begin{bmatrix} \underset{\sim}{w} \\ -1 \end{bmatrix}$ from Theorem 0.3.

Solution

Solution 0.4. Here $\underset{\sim}{w} = (3, -1)$ and $b = 2$. By Theorem 0.3, the graph of f is a hyperplane in $\mathbb{R}^3$ with normal vector $(3, -1, -1)$: it holds the points (x_1, x_2, y) with $3x_1 - x_2 - y + 2 = 0$.

Definition 0.2 (Open half-space)**.** Let $\mathcal{H} = \{\underset{\sim}{x} \in \mathbb{R}^p : \underset{\sim}{w} \cdot \underset{\sim}{x} + b = 0\}$ be a hyperplane (Definition 10 in Linear and Affine Maps). The two **open half-spaces** [2] of $\mathcal{H}$ are

$$\mathcal{H}^+ \stackrel{\text{def}}{=} \{\underset{\sim}{x} \in \mathbb{R}^p : \underset{\sim}{w} \cdot \underset{\sim}{x} + b > 0\} \quad \text{and} \quad \mathcal{H}^- \stackrel{\text{def}}{=} \{\underset{\sim}{x} \in \mathbb{R}^p : \underset{\sim}{w} \cdot \underset{\sim}{x} + b < 0\}. \tag{1}$$

Definition 0.3 (Closed half-space)**.** The **closed half-spaces** of the hyperplane $\mathcal{H}$ of Definition 0.2 also include $\mathcal{H}$ itself: $\mathcal{H}^+ \cup \mathcal{H}$ and $\mathcal{H}^- \cup \mathcal{H}$.

Remark

Remark. Replacing $(\underset{\sim}{w}, b)$ by $(-\underset{\sim}{w}, -b)$ describes the same hyperplane but swaps the names $\mathcal{H}^+$ and $\mathcal{H}^-$. The sign of $\underset{\sim}{w} \cdot \underset{\sim}{x} + b$ tells you which side of the hyperplane $\underset{\sim}{x}$ is on. A linear classifier uses this sign to assign one of two labels.

Example 0.5 (Half-spaces of a line in the plane)**.** Let $\mathcal{H}$ be the line in $\mathbb{R}^2$ with $\underset{\sim}{w} = (1, 2)$ and $b = -2$ (Definition 10 in Linear and Affine Maps), so $\mathcal{H} = \{\underset{\sim}{x} : x_1 + 2x_2 - 2 = 0\}$. Evaluate $\underset{\sim}{w} \cdot \underset{\sim}{x} + b$ at three points (Definition 0.2):

- At $(0, 0)$ it equals $0 + 0 - 2 = -2 < 0$, so $(0, 0) \in \mathcal{H}^-$.
- At $(2, 0)$ it equals $2 + 0 - 2 = 0$, so $(2, 0) \in \mathcal{H}$.
- At $(2, 1)$ it equals $2 + 2 - 2 = 2 > 0$, so $(2, 1) \in \mathcal{H}^+$.

Moving from $(2, 0)$ in the direction $\underset{\sim}{w} = (1, 2)$ gives $(3, 2)$, where the value is $3 + 4 - 2 = 5 > 0$. So the direction $\underset{\sim}{w}$ leads into $\mathcal{H}^+$.

Exercise 0.5 (Finding which open half-space holds a point)**.** The sign of $\underset{\sim}{w} \cdot \underset{\sim}{x} + b$ tells which open half-space of a hyperplane a point $\underset{\sim}{x}$ lies in. Let $\mathcal{H}$ be the line in $\mathbb{R}^2$ with $\underset{\sim}{w} = (2, -1)$ and $b = -1$.
Decide whether $(0, 0)$ is in $\mathcal{H}^+$ or in $\mathcal{H}^-$.

Hint

Evaluate $\underset{\sim}{w} \cdot \underset{\sim}{x} + b$ at $(0, 0)$.

Solution

Solution 0.5. At $(0, 0)$, $\underset{\sim}{w} \cdot \underset{\sim}{x} + b = 0 + 0 - 1 = -1 < 0$, so $(0, 0)$ is in the open half-space $\mathcal{H}^-$.

Exercise 0.6 (Testing whether a point is in a closed half-space)**.** A closed half-space adds the hyperplane itself to an open half-space. Let $\mathcal{H}$ be the line in $\mathbb{R}^2$ with $\underset{\sim}{w} = (2, -1)$ and $b = -3$.
Decide whether the point $(2, 1)$ is in the closed half-space $\mathcal{H}^+ \cup \mathcal{H}$.

Hint

Evaluate $\underset{\sim}{w} \cdot \underset{\sim}{x} + b$ at $(2, 1)$.

Solution

Solution 0.6. At $(2,1)$, $\underset{\sim}{w} \cdot \underset{\sim}{x} + b = 4 - 1 - 3 = 0$, so $(2,1) \in \mathcal{H}$. It is therefore in the closed half-space $\mathcal{H}^+ \cup \mathcal{H}$, though not in the open half-space $\mathcal{H}^+$.

Remark (The two sides of a hyperplane)

Remark 0.1 (The two sides of a hyperplane). Every point of $\mathbb{R}^p$ is in exactly one of $\mathcal{H}^+$, $\mathcal{H}$ and $\mathcal{H}^-$, because the number $\underset{\sim}{w} \cdot \underset{\sim}{x} + b$ is positive, zero or negative. The vector $\underset{\sim}{w}$ points into $\mathcal{H}^+$: the number $\underset{\sim}{w} \cdot \underset{\sim}{x} + b$ goes up when $\underset{\sim}{x}$ moves in the direction $\underset{\sim}{w}$.

Theorem 0.4 (Differences of points in a hyperplane and in a half-space)**.** *Let $\underset{\sim}{w} \in \mathbb{R}^p$ be nonzero, let $b \in \mathbb{R}$, and let $\mathcal{H}$, $\mathcal{H}^+$ be the hyperplane and open half-space of Definition 0.2. For a set $S \subseteq \mathbb{R}^p$, let $D(S) \stackrel{def}{=} \{\underset{\sim}{y} - \underset{\sim}{x} : \underset{\sim}{x}, \underset{\sim}{y} \in S\}$ be the set of differences of its points.*

1. *$D(\mathcal{H}) = \{\underset{\sim}{s} : \underset{\sim}{w} \cdot \underset{\sim}{s} = 0\}$, a subspace of dimension $p - 1$.*
2. *$D(\mathcal{H}^+) = \mathbb{R}^p$, a subspace of dimension p.*

The same holds for $\mathcal{H}^-$.

Proof

Proof. **Part 1.** Let $\underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{H}$. By Theorem 0.1 part 3, $\underset{\sim}{w} \cdot (\underset{\sim}{y} - \underset{\sim}{x}) = 0$, so $\underset{\sim}{y} - \underset{\sim}{x} \in \mathcal{H}_0 = \{\underset{\sim}{s} : \underset{\sim}{w} \cdot \underset{\sim}{s} = 0\}$. Conversely, take any $\underset{\sim}{x}_0 \in \mathcal{H}$ (Theorem 0.1 part 2 gives such a point). For any $\underset{\sim}{s} \in \mathcal{H}_0$, the point $\underset{\sim}{x}_0 + \underset{\sim}{s}$ is in $\mathcal{H}$ by the same part, and $(\underset{\sim}{x}_0 + \underset{\sim}{s}) - \underset{\sim}{x}_0 = \underset{\sim}{s}$. So $D(\mathcal{H}) = \mathcal{H}_0$, which has dimension $p - 1$ by Theorem 0.1 part 1.
Part 2. Every difference is in $\mathbb{R}^p$, so $D(\mathcal{H}^+) \subseteq \mathbb{R}^p$. For the reverse inclusion, take any $\underset{\sim}{x}_0 \in \mathcal{H}$. Take any $\underset{\sim}{v} \in \mathbb{R}^p$. For a number $t > 0$ to be chosen, let $\underset{\sim}{x}_t \stackrel{\text{def}}{=} \underset{\sim}{x}_0 + t\,\underset{\sim}{w}$. Then

$$\begin{aligned} \underset{\sim}{w} \cdot \underset{\sim}{x}_t + b &= (\underset{\sim}{w} \cdot \underset{\sim}{x}_0 + b) + t\,\underset{\sim}{w} \cdot \underset{\sim}{w} && \text{(linearity of } \underset{\sim}{x} \mapsto \underset{\sim}{w} \cdot \underset{\sim}{x}) \\ &= t\,\underset{\sim}{w} \cdot \underset{\sim}{w}, && (\underset{\sim}{x}_0 \in \mathcal{H}) \end{aligned}$$

which is positive, so $\underset{\sim}{x}_t \in \mathcal{H}^+$. Next,

$$\begin{aligned} \underset{\sim}{w} \cdot (\underset{\sim}{x}_t + \underset{\sim}{v}) + b &= (\underset{\sim}{w} \cdot \underset{\sim}{x}_t + b) + \underset{\sim}{w} \cdot \underset{\sim}{v} && \text{(linearity of } \underset{\sim}{x} \mapsto \underset{\sim}{w} \cdot \underset{\sim}{x}) \\ &= t\,\underset{\sim}{w} \cdot \underset{\sim}{w} + \underset{\sim}{w} \cdot \underset{\sim}{v}. && \text{(the display above)} \end{aligned}$$

Because $\underset{\sim}{w} \cdot \underset{\sim}{w} > 0$, this is positive once $t > -\underset{\sim}{w} \cdot \underset{\sim}{v} / \underset{\sim}{w} \cdot \underset{\sim}{w}$. Choose such a $t > 0$. Then $\underset{\sim}{x}_t$

and $\underset{\sim}{x}_t + \underset{\sim}{v}$ are both in $\mathcal{H}^+$, and their difference is $\underset{\sim}{v}$. So $\underset{\sim}{v} \in D(\mathcal{H}^+)$. The whole space $\mathbb{R}^p$ is a subspace of dimension p (Definition 4 in Bases and Dimension).
The proof for $\mathcal{H}^-$ is the same with $-\underset{\sim}{w}$ and $-b$ in place of $\underset{\sim}{w}$ and b. □

Example 0.6 (Two iris species on either side of a line)**.** The `iris` data set in R has the petal length and petal width, in centimeters, of 150 irises. Take the 100 flowers of the species versicolor and virginica, and let $\underset{\sim}{x} = (x_1, x_2)$ be the petal length and petal width of one flower. A logistic regression of the species on $\underset{\sim}{x}$ gives a coefficient vector $\underset{\sim}{w}$ and an intercept b. The line $\underset{\sim}{w} \cdot \underset{\sim}{x} + b = 0$ is a hyperplane in $\mathbb{R}^2$ (Definition 10 in Linear and Affine Maps), and the two species mostly fall in its two open half-spaces (Definition 0.2). Figure 2 shows the line and the flowers.

```
two_species <- dplyr::filter(iris, Species != "setosa") |> droplevels()
fit <- glm(
  Species ~ Petal.Length + Petal.Width,
  family = binomial,
  data = two_species
)
w <- coef(fit)[c("Petal.Length", "Petal.Width")]
b <- coef(fit)[["(Intercept)"]]
two_species$score <- as.vector(
  as.matrix(two_species[c("Petal.Length", "Petal.Width")]) %*% w + b
)
two_species$side <- ifelse(two_species$score > 0, "positive", "negative")
ggplot2::ggplot(two_species) +
  ggplot2::aes(x = Petal.Length, y = Petal.Width, colour = Species) +
  ggplot2::geom_point() +
  ggplot2::geom_abline(intercept = -b / w[[2]], slope = -w[[1]] / w[[2]]) +
  ggplot2::labs(x = "Petal length (cm)", y = "Petal width (cm)")
```

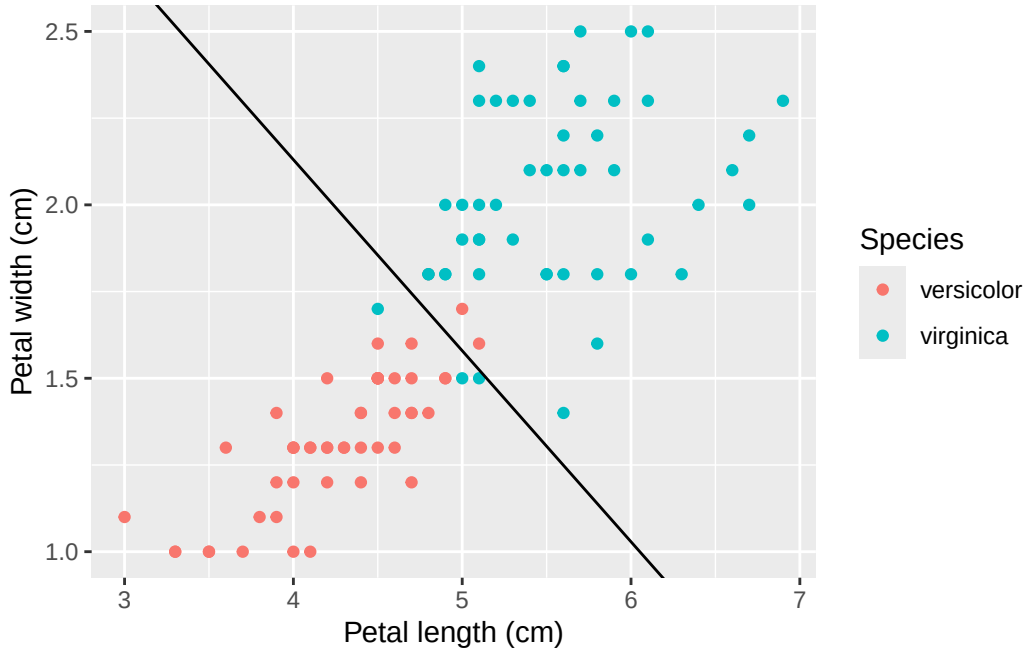

Figure 2: Petal length and width of versicolor and virginica irises, with the line where a logistic regression is indifferent between the two species

```
table(two_species$Species, two_species$side)
#>
#>              negative positive
#>   versicolor       47        3
#>   virginica         3       47
```

The vector $\underset{\sim}{w}$ points toward the virginica side, so the positive half-space $\mathcal{H}^+$ holds 50 of the 100 flowers, and 47 of them are virginica. The line is a set of dimension $p-1=1$, and each side is a region of the plane of dimension $p=2$ (Theorem 0.4).

Exercise 0.7 (Which side of the plane?)**.** Let $\mathcal{H}$ be the plane in $\mathbb{R}^3$ with $\underset{\sim}{w}=(1,1,1)$ and $b=-1$ (Example 0.2).

1. For each point, say whether it is in $\mathcal{H}^+$, $\mathcal{H}$ or $\mathcal{H}^-$ (Definition 0.2): $(0,0,0)$, $(1,0,0)$ and $(1,1,1)$.
2. Write the plane and its half-spaces using $-\underset{\sim}{w}$ and $-b$. Which points from part 1 change sides?
3. State the dimension of $D(\mathcal{H})$ and of $D(\mathcal{H}^+)$ (Theorem 0.4).

Solution

Solution 0.7.

1. Compute $\underset{\sim}{w} \cdot \underset{\sim}{x} + b = x_1 + x_2 + x_3 - 1$ at each point.
   - At $(0, 0, 0)$ it equals $0 + 0 + 0 - 1 = -1 < 0$, so the point is in $\mathcal{H}^-$.
   - At $(1, 0, 0)$ it equals $1 + 0 + 0 - 1 = 0$, so the point is in $\mathcal{H}$.
   - At $(1, 1, 1)$ it equals $1 + 1 + 1 - 1 = 2 > 0$, so the point is in $\mathcal{H}^+$.
2. With $-\underset{\sim}{w} = (-1, -1, -1)$ and $-b = 1$, the equation is $-x_1 - x_2 - x_3 + 1 = 0$. Multiplying both sides by -1 gives the original equation, so it is the same plane. The new expression is the negative of the old one at every point, so the new positive half-space is the old $\mathcal{H}^-$ and the new negative half-space is the old $\mathcal{H}^+$. The origin $(0, 0, 0)$ is now in the positive half-space, and $(1, 1, 1)$ is now in the negative half-space. The point $(1, 0, 0)$ stays on the plane. So the origin and $(1, 1, 1)$ both change names, and the point on the plane does not.
3. Here $p = 3$. $D(\mathcal{H})$ is the subspace $\{\underset{\sim}{s} : s_1 + s_2 + s_3 = 0\}$, which has dimension $p - 1 = 2$. $D(\mathcal{H}^+) = \mathbb{R}^3$ has dimension $p = 3$.

[1] Wikipedia contributors, "Affine space — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Affine_space

[2] Wikipedia contributors, "Half-space (geometry) — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Half-space_(geometry)