

Induction and Recursion

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Remark

Remark. A [proof by induction](#) proves a [proposition](#) about every [integer](#) from some starting integer on. A [sequence](#) defined by a rule on its earlier [terms](#) is the usual place to use it. This page defines such [sequences](#), then adds the stronger form, [strong induction](#), that they often need.

Recursively defined sequences

Definition 0.1 (Recursively defined sequence). Let n_0 be an [integer](#) and m a positive integer. A [sequence](#) $a_{n_0}, a_{n_0+1}, \dots$ is **recursively defined** [1] by two things:

1. **Initial terms:** specific [values](#) for $a_{n_0}, \dots, a_{n_0+m-1}$.
2. **Recurrence:** a rule that gives a_n from the m entries $a_{n-1}, \dots, a_{n-m}$ just before it, for every integer $n \geq n_0 + m$.

The initial terms and the recurrence together fix every [term](#): the first m entries are given, and each later entry follows from the m entries before it.

Example 0.1 (Powers of 3). The [sequence](#) $a_0, a_1, \dots$ with initial [term](#) $a_0 = 1$ and recurrence $a_n = 3a_{n-1}$ for $n \geq 1$ has $m = 1$, and its [terms](#) are 1, 3, 9, 27, ... A [proof by induction](#) shows $a_n = 3^n$ for every [integer](#) $n \geq 0$, using the recurrence in the inductive step: if $a_{n-1} = 3^{n-1}$, then $a_n = 3 \cdot 3^{n-1} = 3^n$.

Definition 0.2 (Fibonacci sequence). The **Fibonacci sequence** $F_1, F_2, \dots$ [2] is the [recursively defined sequence](#) with $m = 2$, initial [terms](#) $F_1 = 1$ and $F_2 = 1$, and recurrence $F_n = F_{n-1} + F_{n-2}$ for every [integer](#) $n \geq 3$.

Example 0.2 (The first Fibonacci numbers). $F_3 = F_2 + F_1 = 1 + 1 = 2$, $F_4 = F_3 + F_2 = 2 + 1 = 3$, $F_5 = F_4 + F_3 = 3 + 2 = 5$, and $F_6 = F_5 + F_4 = 5 + 3 = 8$. So the sequence begins 1, 1, 2, 3, 5, 8, ...

Strong induction

Definition 0.3 (Proof by strong induction). Let n_0 be an integer, and let $P(n)$ be a predicate about the integers $n \geq n_0$. A **proof by strong induction** [3] on n shows that $P(n)$ holds for every integer $n \geq n_0$ by showing one thing: for every integer $n \geq n_0$, if $P(k)$ holds for every integer k with $n_0 \leq k < n$, then $P(n)$ holds.

The assumption that $P(k)$ holds for every integer k with $n_0 \leq k < n$ is the **strong induction hypothesis**.

For $n = n_0$ no integer k satisfies $n_0 \leq k < n$, so the strong induction hypothesis is **vacuously true**, and the proof for $n = n_0$ is a **base case**. When the proof for n needs particular earlier cases that do not exist for the first few n , those few cases are shown separately as further **base cases**.

Example 0.3 (Every term of the Fibonacci sequence is less than a power of 2). Let $F_1, F_2, \dots$ be the Fibonacci sequence. To prove $F_n < 2^n$ for every $n \in \mathbb{N}$, let $P(n)$ be the inequality $F_n < 2^n$ and use Definition 0.3 with $n_0 = 1$.

Base cases, $n = 1$ and $n = 2$. The step below uses the two entries F_{n-1} and F_{n-2} before F_n , which do not exist for $n = 1$ and $n = 2$, so these two cases are shown directly:

$$\begin{aligned} F_1 &= 1 < 2 = 2^1 & (1 < 2) \\ F_2 &= 1 < 4 = 2^2 & (1 < 4) \end{aligned}$$

Step of the strong induction, $n \geq 3$. Let $n \geq 3$, and suppose $P(k)$ holds for every integer k with $1 \leq k < n$. Then $k = n - 1$ and $k = n - 2$ both satisfy $1 \leq k < n$, so $F_{n-1} < 2^{n-1}$ and $F_{n-2} < 2^{n-2}$. Therefore

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} & (\text{definition of } F_n, \text{ since } n \geq 3) \\ &< 2^{n-1} + 2^{n-2} & (\text{strong induction hypothesis, twice}) \\ &< 2^{n-1} + 2^{n-1} & (2^{n-2} < 2^{n-1}) \\ &= 2 \cdot 2^{n-1} & (a + a = 2a) \\ &= 2^n, & (2 \cdot 2^{n-1} = 2^n) \end{aligned}$$

which is $P(n)$. By Definition 0.3, $F_n < 2^n$ for every $n \in \mathbb{N}$.

Further reading

- Velleman [4] is a structured introduction to proof techniques, including [proof by induction](#).

References

- [1] Wikipedia contributors, “Recurrence relation — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Recurrence_relation
- [2] Wikipedia contributors, “Fibonacci sequence — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Fibonacci_sequence
- [3] Wikipedia contributors, “Mathematical induction — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Mathematical_induction
- [4] D. J. Velleman, *How to prove it: A structured approach*, 3rd ed. Cambridge University Press, 2019. doi: [10.1017/9781108539890](https://doi.org/10.1017/9781108539890).