Taylor Series

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Taylor polynomials and series

Definition 0.1 (Taylor polynomial (Taylor expansion))**.** Let f be a function that has at least n derivatives at a point x_0. The **Taylor polynomial** (also called the **Taylor expansion**) of degree n for f centered at x_0 is:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned} \tag{1}$$

Example 0.1 (Taylor series of the exponential function)**.** For $f(x) = e^x$ centered at $x_0 = 0$, every derivative is

$$\begin{aligned} f^{(k)}(0) &= e^0 \\ &= 1. \end{aligned}$$

The degree-n Maclaurin polynomial is:

$$\begin{aligned} P_n(x) &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \\ &= \sum_{k=0}^{n} \frac{x^k}{k!} \end{aligned}$$

Because the remainder $R_n(x) \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$, the Taylor series converges everywhere to e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Definition 0.2 (Maclaurin polynomial)**.** When $x_0 = 0$, the Taylor polynomial $P_n(x)$ (Definition 0.1) is called the **Maclaurin polynomial**.

Definition 0.3 (Taylor series)**.** Let f and x_0 be as in Definition 0.1. When f is infinitely differentiable and the series converges to $f(x)$ on an open interval containing x_0, the infinite sum $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ is the **Taylor series** of f centered at x_0.

Video

Chapter 11 of the *Essence of calculus* series by 3Blue1Brown develops the geometric and polynomial intuition behind Taylor approximations:

- Taylor series shows how higher-order derivatives match the curvature and rate of change of a function near a point to build polynomial approximations (Definition 0.1).

Polynomials, exponentials and trigonometric functions

Exercise 0.1 (Taylor polynomial of a polynomial at the origin)**.** Adapted from Miller [1], Question 1.1.16.
Find the first five terms (up to degree 4) of the Taylor series for

$$f(x) = x^8 + x^4 + 3$$

at $x_0 = 0$.

Solution

Solution 0.1. Evaluate f and its first four derivatives at $x = 0$:

$$\begin{aligned} f(x) &= x^8 + x^4 + 3 && \implies f(0) = 3 \\ f'(x) &= 8x^7 + 4x^3 && \implies f'(0) = 0 \\ f''(x) &= 56x^6 + 12x^2 && \implies f''(0) = 0 \\ f'''(x) &= 336x^5 + 24x && \implies f'''(0) = 0 \\ f^{(4)}(x) &= 1680x^4 + 24 && \implies f^{(4)}(0) = 24 \end{aligned}$$

By Equation 1:

$$\begin{aligned} P_4(x) &= 3 + 0x + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 \\ &= 3 + x^4 \end{aligned}$$

Because f is already a polynomial centered at 0, its Taylor polynomial of degree 4 simply collects its terms of degree at most 4.

Exercise 0.2 (Taylor expansion of a polynomial at a nonzero point)**.** Adapted from Miller [1], Question 1.1.17.
Find the first three terms (up to degree 2) of the Taylor series for

$$f(x) = x^8 + x^4 + 3$$

at $x_0 = 1$.

Solution

Solution 0.2. Evaluate f and its first two derivatives at $x_0 = 1$:

$$\begin{aligned} f(1) &= 1^8 + 1^4 + 3 \\ &= 5 \\ f'(1) &= 8(1)^7 + 4(1)^3 \\ &= 12 \\ f''(1) &= 56(1)^6 + 12(1)^2 \\ &= 68 \end{aligned}$$

The degree-2 Taylor polynomial is:

$$\begin{aligned} P_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 5 + 12(x-1) + 34(x-1)^2 \end{aligned}$$

Example 0.1 (Taylor series of the cosine function)**.** For $f(x) = \cos x$ centered at $x_0 = 0$, the derivatives follow a repeating cycle of length 4:

$$
\begin{aligned}
f(0) &= \cos(0) \\
&= 1, \\
f'(0) &= -\sin(0) \\
&= 0, \\
f''(0) &= -\cos(0) \\
&= -1, \\
f'''(0) &= \sin(0) \\
&= 0,
\end{aligned}
$$

and in general $f^{(2k)}(0) = (-1)^k$ and $f^{(2k+1)}(0) = 0$ for all $k \geq 0$. The Taylor series (Maclaurin series) converges everywhere to $\cos x$:

$$
\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}
\end{aligned}
$$

Example 0.2 (Taylor series of the sine function)**.** For $f(x) = \sin x$ centered at $x_0 = 0$, the derivatives evaluate at 0 to $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$ for all $k \geq 0$. The Taylor series converges everywhere to $\sin x$:

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}
\end{aligned}
$$

Exercise 0.3 (Taylor expansion of a scaled cosine)**.** Adapted from Miller [1], Question 1.1.18.
Find the terms up to degree 4 in the Taylor series for

$$f(x) = \cos(5x)$$

at $x_0 = 0$.

Solution

Solution 0.3. Recall the standard Maclaurin series for cosine:

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \ldots$$

Substitute $u = 5x$:

$$\begin{aligned}\cos(5x) &= 1 - \frac{(5x)^2}{2!} + \frac{(5x)^4}{4!} - \ldots \\ &= 1 - \frac{25}{2}x^2 + \frac{625}{24}x^4 - \ldots\end{aligned}$$

Remark: Substituting into known Taylor expansions avoids computing high-order derivatives by hand.

Exercise 0.4 (Taylor series of a power of a function)**.** Adapted from Miller [1], Question 1.1.19.
Find the terms up to degree 4 in the Taylor series for

$$f(x) = \cos^3(5x)$$

at $x_0 = 0$.

Solution

Solution 0.4. From Exercise 0.3, the expansion of $\cos(5x)$ up to order x^4 is:

$$\cos(5x) = 1 - \frac{25}{2}x^2 + \frac{625}{24}x^4 + O(x^6)$$

Cube this expansion using $(1-y)^3 = 1 - 3y + 3y^2 - y^3$ with $y = \frac{25}{2}x^2 - \frac{625}{24}x^4$:

$$\begin{aligned}\cos^3(5x) &= 1 - 3\left(\frac{25}{2}x^2 - \frac{625}{24}x^4\right) + 3\left(\frac{25}{2}x^2\right)^2 + O(x^6) \\ &= 1 - \frac{75}{2}x^2 + \frac{625}{8}x^4 + \frac{1875}{4}x^4 + O(x^6) \\ &= 1 - \frac{75}{2}x^2 + \frac{4375}{8}x^4 + O(x^6)\end{aligned}$$

Exercise 0.5 (Linear Taylor polynomial of the exponential)**.** Adapted from Miller [1], Question 1.1.20.
Find the first two terms (degree 1) of the Taylor series for

$$f(x) = e^x$$

at $x_0 = 0$.

Solution

Solution 0.5. Since

$$\begin{aligned} f(0) &= e^0 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f'(0) &= e^0 \\ &= 1 : \end{aligned}$$

$$\begin{aligned} P_1(x) &= f(0) + f'(0)x \\ &= 1 + x \end{aligned}$$

The full Taylor series is

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots . \end{aligned}$$

Exercise 0.6 (Taylor series of a composite exponential)**.** Adapted from Miller [1], Question 1.1.21.
Find the first six terms (up to degree 5) of the Taylor series for

$$f(x) = \exp\{x^8\}$$

at $x_0 = 0$.

Solution

Solution 0.6. Substitute $u = x^8$ into $e^u = 1 + u + \frac{u^2}{2!} + \dots$:

$$\exp\{x^8\} = 1 + x^8 + \frac{x^{16}}{2} + \dots$$

Because the lowest-order variable term is x^8, all coefficients for powers x^1, x^2, x^3, x^4, x^5 are zero:

$$\begin{aligned} P_5(x) &= 1 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5 \\ &= 1 \end{aligned}$$

Roots, logarithms and compound functions

Exercise 0.1 (Taylor expansion of the normal density)**.** Adapted from Miller [1], Question 1.1.22.
Find the terms up to degree 3 of the Taylor series for the standard normal density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$$

at $x_0 = 0$.

Solution

Solution 0.1. Using $e^u = 1 + u + O(u^2)$ with $u = -x^2/2$:

$$\exp\left\{-\frac{x^2}{2}\right\} = 1 - \frac{x^2}{2} + O(x^4)$$

Multiplying by the normalization constant $1/\sqrt{2\pi}$:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} - \frac{x^2}{2\sqrt{2\pi}} + O(x^4)$$

Up to degree 3, the odd powers vanish by symmetry, so:

$$P_3(x) = \frac{1}{\sqrt{2\pi}} - \frac{x^2}{2\sqrt{2\pi}}$$

Theorem 0.1 (Maclaurin polynomial of the standard normal density)**.** *The Maclaurin polynomial (Definition 0.2) of degree 3 of the standard normal density* $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$ *is*

$$P_3(x) = \frac{1}{\sqrt{2\pi}} - \frac{x^2}{2\sqrt{2\pi}}$$

Proof

Proof. This is the solution to Exercise 0.1, worked in Solution 0.1. □

Exercise 0.2 (Taylor expansion of the square root)**.** Adapted from Miller [1], Question 1.1.23.

Find the first three terms (up to degree 2) of the Taylor series for

$$f(x) = \sqrt{x}$$

at $x_0 = 1/3$.

Solution

Solution 0.2. Compute the values at $x_0 = 1/3$:

$$\begin{aligned} f(1/3) &= \left(\frac{1}{3}\right)^{1/2} \\ &= \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \\ f'(x) &= \frac{1}{2}x^{-1/2} \\ \implies f'(1/3) &= \frac{1}{2}\sqrt{3} \\ &= \frac{\sqrt{3}}{2} \\ f''(x) &= -\frac{1}{4}x^{-3/2} \\ \implies f''(1/3) &= -\frac{1}{4}\cdot 3\sqrt{3} \\ &= -\frac{3\sqrt{3}}{4} \end{aligned}$$

The degree-2 Taylor polynomial is:

$$P_2(x) = \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\left(x - \frac{1}{3}\right) - \frac{3\sqrt{3}}{8}\left(x - \frac{1}{3}\right)^2$$

Exercise 0.3 (Taylor expansion of a fractional power)**.** Adapted from Miller [1], Question 1.1.24.
Find the first three terms of the Taylor series for

$$f(x) = (1 + x)^{1/3}$$

at $x_0 = 1/2$.

Solution

Solution 0.3. At $x_0 = 1/2$, $1 + x_0 = 3/2$:

$$\begin{aligned} f(1/2) &= \left(\frac{3}{2}\right)^{1/3} \\ f'(x) &= \frac{1}{3}(1+x)^{-2/3} \\ \implies f'(1/2) &= \frac{1}{3}\left(\frac{3}{2}\right)^{-2/3} \\ &= \frac{1}{3}\left(\frac{2}{3}\right)^{2/3} \\ f''(x) &= -\frac{2}{9}(1+x)^{-5/3} \\ \implies f''(1/2) &= -\frac{2}{9}\left(\frac{3}{2}\right)^{-5/3} \\ &= -\frac{2}{9}\left(\frac{2}{3}\right)^{5/3} \end{aligned}$$

Dividing $f''(1/2)$ by 2! gives $-\frac{1}{9}\left(\frac{2}{3}\right)^{5/3}$. The Taylor polynomial is:

$$P_2(x) = \left(\frac{3}{2}\right)^{1/3} + \frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\left(x - \frac{1}{2}\right) - \frac{1}{9}\left(\frac{2}{3}\right)^{5/3}\left(x - \frac{1}{2}\right)^2$$

Exercise 0.4 (Taylor expansion of $x \log x$)**.** Adapted from Miller [1], Question 1.1.25. Find the first three terms (up to degree 2) of the Taylor series for

$$f(x) = x \log x$$

at $x_0 = 1$.

Solution

Solution 0.4. Evaluate derivatives at $x_0 = 1$:

$$\begin{aligned} f(1) &= 1 \log 1 \\ &= 0 \\ f'(x) &= \log x + 1 \\ \implies f'(1) &= 0 + 1 \\ &= 1 \\ f''(x) &= \frac{1}{x} \implies f''(1) = 1 \end{aligned}$$

Therefore:

$$\begin{aligned} P_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= (x-1) + \frac{1}{2}(x-1)^2 \end{aligned}$$

Exercise 0.5 (Maclaurin series of $\log(1+x)$)**.** Adapted from Miller [1], Question 1.1.26. Find the first three terms of the Taylor series for

$$f(x) = \log(1+x)$$

at $x_0 = 0$.

Solution

Solution 0.5. Differentiating repeatedly at $x_0 = 0$:

$$\begin{aligned} f(0) &= \log 1 \\ &= 0 \\ f'(x) &= (1+x)^{-1} \implies f'(0) = 1 \\ f''(x) &= -(1+x)^{-2} \implies f''(0) = -1 \\ f'''(x) &= 2(1+x)^{-3} \implies f'''(0) = 2 \end{aligned}$$

The Taylor polynomial of degree 3 is:

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Theorem 0.2 (Maclaurin polynomial of $\log(1+x)$)**.** *The Maclaurin polynomial (Definition 0.2) of degree 3 of* $f(x) = \log(1+x)$ *is*

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Proof

Proof. This is the solution to Exercise 0.5, worked in Solution 0.5. □

Exercise 0.6 (Maclaurin series of $\log(1-x)$)**.** Adapted from Miller [1], Question 1.1.27. Find the terms up to degree 3 of the Taylor series for

$$f(x) = \log(1-x)$$

at $x_0 = 0$.

Solution

Solution 0.6. Substitute $u = -x$ into the expansion for $\log(1+u)$ from Theorem 0.2:

$$\begin{aligned}\log(1-x) &= (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \ldots \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\end{aligned}$$

Theorem 0.3 (Maclaurin polynomial of $\log(1-x)$)**.** *The Maclaurin polynomial (Definition 0.2) of degree 3 of* $f(x) = \log(1-x)$ *is*

$$P_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

Proof

Proof. This is the solution to Exercise 0.6, worked in Solution 0.6. □

Exercise 0.7 (Taylor series of $\log((1-x)e^x)$)**.** Adapted from Miller [1], Question 1.1.28. Find the first two non-zero terms of the Taylor series for

$$f(x) = \log((1-x)e^x)$$

at $x_0 = 0$.

Solution

Solution 0.7. Expand using logarithm properties:

$$\begin{aligned}f(x) &= \log(1-x) + \log(e^x) \\ &= \log(1-x) + x\end{aligned}$$

Substitute the series for $\log(1-x)$ from Theorem 0.3:

$$f(x) = \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - ...\right) + x$$
$$= -\frac{x^2}{2} - \frac{x^3}{3} - ...$$

The constant and linear terms vanish ($f(0) = 0$, $f'(0) = 0$). The first two non-zero terms are:

$$-\frac{x^2}{2} - \frac{x^3}{3}$$

Exercise 0.8 (Taylor series of a product of known expansions)**.** Adapted from Miller [1], Question 1.1.29.
Find the terms up to degree 2 of the Taylor series for

$$f(x) = \cos(x)\log(1 + x)$$

at $x_0 = 0$.

Solution

Solution 0.8. Multiply the known Maclaurin expansions:

$$\cos x = 1 - \frac{x^2}{2} + O(x^4)$$
$$\log(1 + x) = x - \frac{x^2}{2} + O(x^3)$$

Multiplying and dropping terms of order x^3 and higher:

$$f(x) = \left(1 - \frac{x^2}{2} + ...\right)\left(x - \frac{x^2}{2} + ...\right)$$
$$= x - \frac{x^2}{2} + O(x^3)$$

Exercise 0.9 (Taylor series of $\log(1 + 2x)$)**.** Adapted from Miller [1], Question 1.1.30.
Find the first two non-zero terms of the Taylor series for

$$f(x) = \log(1 + 2x)$$

at $x_0 = 0$.

Solution

Solution 0.9. Substitute $u = 2x$ into the Maclaurin series $\log(1+u) = u - \frac{u^2}{2} + ...$:

$$\begin{aligned} \log(1+2x) &= (2x) - \frac{(2x)^2}{2} + ... \\ &= 2x - 2x^2 + ... \end{aligned}$$

[1] S. J. Miller, *The probability lifesaver: Calculus review problems*. 2016. Available: https://web.williams.edu/Mathematics/sjmiller/public_html/probabilitylifesaver/index.htm#:~:text=http%3A//web.williams.edu/Mathematics/sjmiller/public_html/probabilitylifesaver/supplementalchap_calcreview.pdf