

Integrals and the Fundamental Theorem of Calculus

Last modified: 2026-10-27 20:53:54 (PDT)

Integration

Integration is the inverse operation of differentiation: it recovers a function from its derivative and accumulates quantities such as areas, totals, and probabilities. We begin with antiderivatives, then state basic integration rules, and conclude with the Fundamental Theorem of Calculus and a worked example from probability.

Videos

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist introduces integral calculus:

- [Intro to Integral Calculus](#)
- [What Integral Calculus Is](#)

Antiderivatives

Definition 0.1 (Antiderivative). A function F is an **antiderivative** of f on an interval I [1] if:

$$\frac{\partial}{\partial x} F(x) = f(x), \quad \forall x \in I$$

Finding an antiderivative of f is called **antidifferentiation**, or **antidifferentiating** f . [2, Sec. 4.1], pp. 248–249

Definition 0.2 (Indefinite integral). The **indefinite integral** of f [1] is the family of all antiderivatives (Definition 0.1) of f :

$$\int f(x) dx = F(x) + C$$

where F is any one antiderivative of f and C is an arbitrary constant of integration.
[2, Sec. 4.1], pp. 248–249

Example 0.1 (Antiderivative of x^2). For $f(x) = x^2$, an antiderivative is $F(x) = \frac{x^3}{3}$, since

$$\begin{aligned}\frac{\partial}{\partial x} \frac{x^3}{3} &= x^2 \\ &= f(x).\end{aligned}$$

Adding any constant C gives another antiderivative; for example, with $C = 7$, $F(x) = \frac{x^3}{3} + 7$ also satisfies $F'(x) = x^2$, since adding a constant does not change the derivative. $G(x) = x^3$ is not an antiderivative of x^2 : $G'(x) = 3x^2 \neq x^2$ for $x \neq 0$.
See Figure 1.

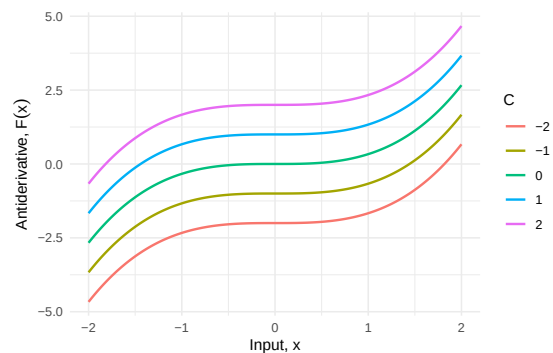
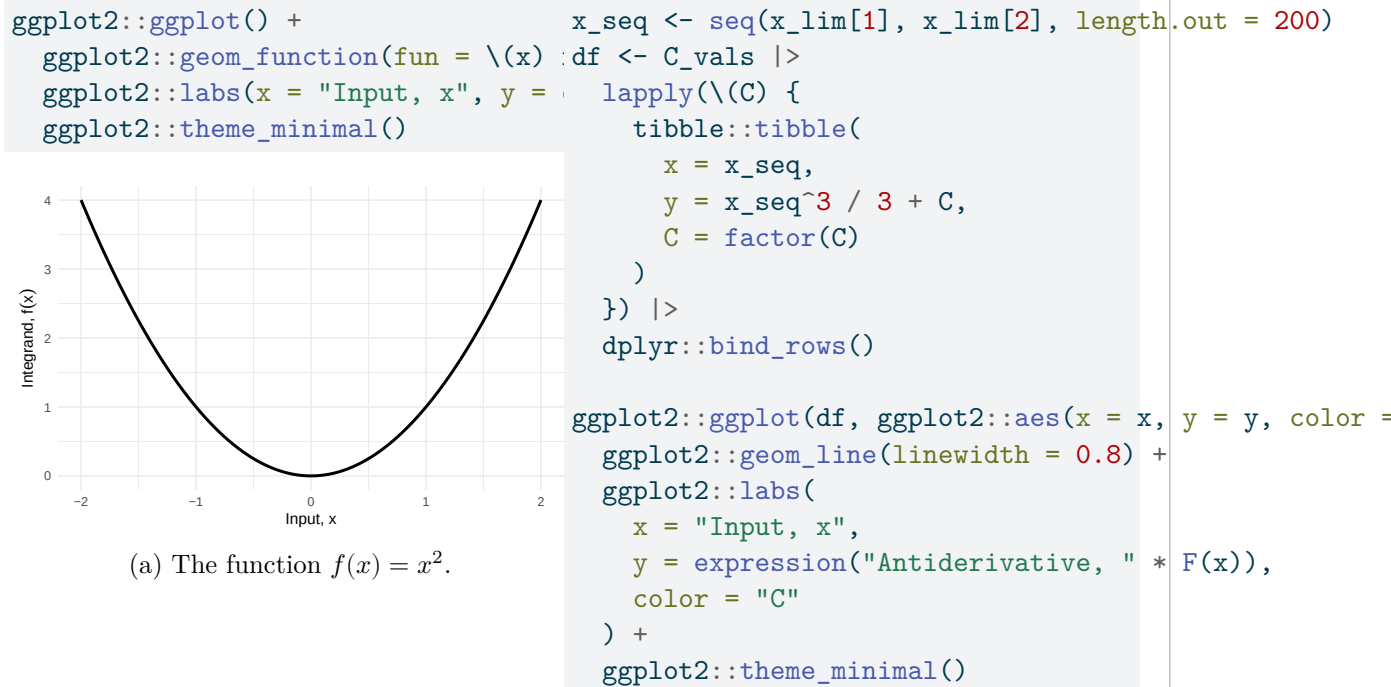


Figure 1: The function $f(x) = x^2$ and five antiderivatives $F(x) = x^3/3 + C$ for $C \in \{-2, -1, 0, 1, 2\}$. Each antiderivative has the same derivative f ; they differ only by a vertical shift.

Theorem 0.1 (Basic integration rules). *Each antiderivative in the table is defined only up to an arbitrary constant C (see Definition 0.1); the table omits $+C$ from every row for brevity.*

Function $f(x)$	Antiderivative $F(x)$	Condition
c	cx	—
x^n	$\frac{x^{n+1}}{n+1}$	$n \neq -1$
$\frac{1}{x}$	$\log\{ x \}$	$x \neq 0$
e^x	e^x	(self-antiderivative)
e^{cx}	$\frac{1}{c}e^{cx}$	$c \neq 0$
$\sin x$	$-\cos x$	—
$\cos x$	$\sin x$	—
$c \cdot f(x)$	$c \cdot F(x)$	—
$f(x) + g(x)$	$F(x) + G(x)$	—

The first two rows, the trigonometric rows ($\sin x$, $\cos x$), and the bottom two rows (linearity) are from [2, Sec. 4.1], p. 250 “Basic Integration Rules”; $1/x$ is from [2, Sec. 5.2], Theorem 5.5, p. 324; e^x and e^{cx} are from [2, Sec. 5.4], Theorem 5.12, p. 346.

Example 0.2 (Antiderivative of $3x^2 - 1$). By the power rule ($n = 2$) and linearity from Theorem 0.1:

$$\begin{aligned}\int (3x^2 - 1) dx &= 3 \cdot \frac{x^3}{3} - x + C \\ &= x^3 - x + C.\end{aligned}$$

Verify by differentiating:

$$\begin{aligned}\frac{\partial}{\partial x}(x^3 - x + C) &= 3x^2 - 1 \\ &= f(x),\end{aligned}$$

as required.

Videos

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist has videos on these rules:

- [The Integral Calculus Rules](#)
- [Indefinite Integral Exercises](#)

Regularity Conditions

Definition 0.3 (Differentiable on an interval). A function f is **differentiable on** an interval if it is differentiable ([Definition 6 in Derivatives and Taylor Series](#)) at every point of the interval other than its **endpoints**, and, at each endpoint that belongs to the interval, it has the one-sided derivative ([Definition 12 in Derivatives and Taylor Series](#)) from inside the interval: the right-hand derivative at the left endpoint and the left-hand derivative at the right endpoint.

[2, Sec. 2.1], p. 100

Example 0.3 (Differentiable on one interval but not another).

- $f(x) = x^2$ is differentiable on $[0, 1]$: the computation of [Example 4 in Derivatives and Taylor Series](#) with 3 replaced by any c gives $\frac{f(c+h)-f(c)}{h} = 2c + h$, which tends to $2c$, including the one-sided limits at the endpoints 0 and 1.
- $g(x) = \sqrt[3]{x}$ is differentiable on $[1, 2]$, but not on $[-1, 1]$: the point 0 is in $[-1, 1]$ and is not an endpoint, and [Example 4 in Derivatives and Taylor Series](#) shows g is not differentiable there.

Definition 0.4 (Continuous function). A function f is **continuous at** $x = c$ [3] if all three conditions hold:

1. $f(c)$ is defined,
2. $\lim_{x \rightarrow c} f(x)$ exists, and
3. $\lim_{x \rightarrow c} f(x) = f(c)$.

[2, Sec. 1.4], p. 73

Definition 0.5 (Discontinuous function). If any of the three conditions of [Definition 0.4](#) fails, f is **discontinuous** at c .

[2, Sec. 1.4], p. 73

Example 0.4 (A continuous function, and one failure of each condition).

- $f(x) = x^2$ is continuous at $c = 1$: $f(1) = 1$ is defined, and

$$\begin{aligned}\lim_{x \rightarrow 1} x^2 &= 1 \\ &= f(1).\end{aligned}$$

- $g(x) = \frac{x^2-1}{x-1}$ fails condition 1 at $c = 1$: $g(1)$ is not defined (it would divide by zero), even though

$$\begin{aligned}\lim_{x \rightarrow 1} g(x) &= \lim_{x \rightarrow 1} (x + 1) \\ &= 2\end{aligned}$$

exists ($x^2 - 1 = (x - 1)(x + 1)$, and the factor $x - 1$ cancels for $x \neq 1$).

- The step function $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$ fails condition 2 at $c = 0$: values to the left are all 0 and values to the right are all 1, so $\lim_{x \rightarrow 0} H(x)$ does not exist.
- $k(x) = x^2$ for $x \neq 1$, with $k(1) = 5$, fails condition 3 at $c = 1$: $\lim_{x \rightarrow 1} k(x) = 1$ exists but differs from $k(1) = 5$.

Definition 0.6 (Jump discontinuity). A function f has a **jump discontinuity** at c [4] if both one-sided limits ([Definition 4 in Derivatives and Taylor Series](#)) $\lim_{x \rightarrow c^-} f(x)$ and $\lim_{x \rightarrow c^+} f(x)$ exist but are not equal. Then $\lim_{x \rightarrow c} f(x)$ does not exist, so f is discontinuous at c ([Definition 0.5](#)).

Example 0.5 (A jump discontinuity, and a discontinuity that is not a jump).

- The step function H of [Example 2 in Derivatives and Taylor Series](#) has a jump discontinuity at 0: $\lim_{x \rightarrow 0^-} H(x) = 0$ and $\lim_{x \rightarrow 0^+} H(x) = 1$ both exist, and $0 \neq 1$. The size of the jump is $1 - 0 = 1$.
- $g(x) = \frac{x^2-1}{x-1}$ of [Example 0.4](#) is discontinuous at 1, because $g(1)$ is not defined, but it does not have a jump discontinuity there: for $x \neq 1$, $g(x) = x + 1$, so both one-sided limits at 1 equal $1 + 1 = 2$.

Definition 0.7 (Continuous on a closed interval). A function f is **continuous on a closed interval** $[a, b]$ if all three conditions hold:

1. f is continuous ([Definition 0.4](#)) at every point of the open interval (a, b) ,
2. $\lim_{x \rightarrow a^+} f(x) = f(a)$, and
3. $\lim_{x \rightarrow b^-} f(x) = f(b)$.

[2, Sec. 1.4], p. 73

Example 0.6 (Continuity on $[0, 1]$ uses one-sided limits at the endpoints). Let $f(x) = \sqrt{x}$, the [square root](#), defined for $x \geq 0$. Because f is undefined for $x < 0$, only the right-hand limit of f at 0 makes sense, and Definition [0.7](#) asks only for that one-sided limit at the endpoint 0. Here f is continuous at every point of $(0, 1)$,

$$\begin{aligned}\lim_{x \rightarrow 0^+} \sqrt{x} &= 0 \\ &= f(0),\end{aligned}$$

and

$$\begin{aligned}\lim_{x \rightarrow 1^-} \sqrt{x} &= 1 \\ &= f(1),\end{aligned}$$

so f is continuous on $[0, 1]$ (Definition [0.7](#)).

Example 0.7 (A function not continuous on $[0, 1]$). Let $f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 2$. Conditions 1 and 2 of Definition [0.7](#) hold, but

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= 0 \\ &\neq 2 \\ &= f(1),\end{aligned}$$

so condition 3 fails and f is not continuous on $[0, 1]$.

Definition 0.8 (Partition of an interval). A **partition** $\mathcal{P}$ of a closed interval $[a, b]$ [5] is a finite list of points

$$\begin{aligned}a &= x_0 \\ &< x_1 \\ &< \dots \\ &< x_n \\ &= b.\end{aligned}$$

It splits $[a, b]$ into the n subintervals $[x_{i-1}, x_i]$, of widths $\Delta x_i \stackrel{\text{def}}{=} x_i - x_{i-1}$. [2, Sec. 4.3]

Example 0.8 (A partition of $[0, 1]$). The points $0 < 0.25 < 0.5 < 1$ form a partition of $[0, 1]$ with $n = 3$ subintervals, of widths $\Delta x_1 = 0.25$, $\Delta x_2 = 0.25$, and $\Delta x_3 = 0.5$.

Example 0.9 (Lists that are not partitions of $[0, 1]$).

- $0, 0.5, 0.25, 1$ is not a partition: the points are not increasing, since $0.5 > 0.25$.
- $0 < 0.5$ is not a partition of $[0, 1]$: its last point is 0.5 , not $b = 1$.

Definition 0.9 (Mesh of a partition). The **mesh** of a partition $\mathcal{P}$ (Definition 0.8) is its largest subinterval width,

$$\|\mathcal{P}\| \stackrel{\text{def}}{=} \max_{i \in \{1, \dots, n\}} \Delta x_i,$$

where n is the number of subintervals of $\mathcal{P}$.

[2, Sec. 4.3]

Example 0.10 (The mesh of a partition of $[0, 1]$). For the partition of $[0, 1]$ in Example 0.8, with widths $\Delta x_1 = 0.25$, $\Delta x_2 = 0.25$, and $\Delta x_3 = 0.5$, the mesh is the largest of these widths, $\|\mathcal{P}\| = 0.5$.

Definition 0.10 (Sample point). For a partition $\mathcal{P}$ of $[a, b]$ (Definition 0.8), choose a **sample point** x_i^* in each subinterval $[x_{i-1}, x_i]$.

[2, Sec. 4.3]

Definition 0.11 (Riemann sum). Let f be a function on $[a, b]$, let $\mathcal{P}$ be a partition

$$\begin{aligned} a &= x_0 \\ &< x_1 \\ &< \dots \\ &< x_n \\ &= b \end{aligned}$$

of $[a, b]$ (Definition 0.8), and choose a sample point x_i^* (Definition 0.10) in each subinterval $[x_{i-1}, x_i]$. The **Riemann sum** of f for $\mathcal{P}$ [6] and these sample points is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i,$$

where $\Delta x_i = x_i - x_{i-1}$.

[2, Sec. 4.3]

Example 0.11 (A Riemann sum for x^2 on $[0, 1]$). Let $f(x) = x^2$, take the partition $0 < 0.25 < 0.5 < 1$ of Example 0.8, with widths $\Delta x_1 = 0.25$, $\Delta x_2 = 0.25$, and $\Delta x_3 = 0.5$, and take each sample point at the right end of its subinterval: $x_1^* = 0.25$, $x_2^* = 0.5$, and $x_3^* = 1$. Then

$$\begin{aligned} \sum_{i=1}^3 f(x_i^*) \Delta x_i &= f(0.25) \cdot 0.25 + f(0.5) \cdot 0.25 + f(1) \cdot 0.5 && \text{(write out the three terms)} \\ &= 0.0625 \cdot 0.25 + 0.25 \cdot 0.25 + 1 \cdot 0.5 && \text{(evaluate } f(x) = x^2 \text{)} \\ &= 0.015625 + 0.0625 + 0.5 && \text{(multiply)} \\ &= 0.578125 && \text{(add)} \end{aligned}$$

Definition 0.12 (Riemann integral (definite integral)). Let f be a **bounded** function on $[a, b]$. For each partition $\mathcal{P}$ of $[a, b]$ (Definition 0.8), choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$. The **Riemann integral** of f over $[a, b]$ [7] (also called the **definite integral** of f from a to b) is the limit of the Riemann sums (Definition 0.11) as the mesh (Definition 0.9) shrinks to zero:

$$\int_a^b f(x) dx \stackrel{\text{def}}{=} \lim_{\|\mathcal{P}\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i,$$

when that limit exists and has the same value for every choice of the partitions and of the sample points x_i^* .

[2, Sec. 4.3], p. 272

Definition 0.13 (Riemann integrable). A bounded function f is **Riemann integrable** on $[a, b]$ if the Riemann sums (Definition 0.11) $\sum_{i=1}^n f(x_i^*) \Delta x_i$, over partitions $\mathcal{P}$ of $[a, b]$ (Definition 0.8) with a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$, approach a real-number limit as the mesh $\|\mathcal{P}\|$ (Definition 0.9) shrinks to zero, and that limit is the same for every choice of the partitions and of the sample points.

[2, Sec. 4.3], p. 272

Example 0.12 (A constant function is integrable). Let $f(x) = 2$ on $[0, 3]$. For every partition and every choice of sample points,

$$\begin{aligned}
\sum_{i=1}^n f(x_i^*) \Delta x_i &= \sum_{i=1}^n 2 \Delta x_i \quad (f \text{ is } 2 \text{ everywhere}) \\
&= 2 \sum_{i=1}^n \Delta x_i \quad (\text{factor out the constant}) \\
&= 2 \cdot (3 - 0) \quad (\text{the widths add up to the length of } [0, 3]) \\
&= 6, \quad (\text{multiply})
\end{aligned}$$

so the sums have the same limit, 6, for every choice: f is Riemann integrable on $[0, 3]$ (Definition 0.13), and $\int_0^3 2 \, dx = 6$ (Definition 0.12).

Definition 0.14 (Integrand). In an integral such as $\int_a^b f(x) \, dx$ (Definition 0.12) or $\int f(x) \, dx$ (Definition 0.2), the function f being integrated is the **integrand**.

Example 0.13 (Integrands).

- In $\int_0^3 2 \, dx = 6$ (Example 0.12), the integrand is the **constant function** $f(x) = 2$, and the **limits of integration** are 0 and 3.
- In $\int (3x^2 - 1) \, dx = x^3 - x + C$ (Example 0.2), the integrand is $f(x) = 3x^2 - 1$; its value at $x = 2$ is $3 \cdot 4 - 1 = 11$.

i Remark (Riemann integrable functions and Riemann integrals)

Remark 0.1 (Riemann integrable functions and Riemann integrals). A bounded function f is Riemann integrable on $[a, b]$ exactly when its Riemann integral $\int_a^b f(x) \, dx$ (Definition 0.12) exists; the integral is the common limit of the sums. For example, let $g(x) = 1$ when x is **rational** and $g(x) = 0$ when x is **irrational**, on $[0, 1]$. Every subinterval contains both rational and irrational points. Choosing every sample point rational gives $\sum_{i=1}^n 1 \cdot \Delta x_i = 1$ for every partition, because the widths Δx_i add up to the length $1 - 0 = 1$ of $[0, 1]$, and choosing every sample point irrational gives $\sum_{i=1}^n 0 \cdot \Delta x_i = 0$. The two limits differ, so g is not Riemann integrable on $[0, 1]$, and $\int_0^1 g(x) \, dx$ does not exist.

Definition 0.15 (Equal-width Riemann sum). For a bounded function f on $[a, b]$ and a positive integer n , split $[a, b]$ into n subintervals of equal width $\Delta x \stackrel{\text{def}}{=} (b - a)/n$, and let x_i^* be any point in the i -th subinterval. The **equal-width Riemann sum** is

$$S_n \stackrel{\text{def}}{=} \sum_{i=1}^n f(x_i^*) \Delta x.$$

Example 0.14 (An equal-width Riemann sum). Let $f(x) = x^2$ on $[0, 1]$, with $n = 2$, so $\Delta x = 1/2$, and take each sample point at the right end of its subinterval: $x_1^* = \frac{1}{2}$ and $x_2^* = 1$. Then

$$\begin{aligned} S_2 &= f\left(\frac{1}{2}\right) \cdot \frac{1}{2} + f(1) \cdot \frac{1}{2} && \text{(equal-width Riemann sum with } n = 2\text{)} \\ &= \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} && \text{(evaluate } f(x) = x^2\text{)} \\ &= \frac{5}{8} && \text{(add)} \end{aligned}$$

Videos

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist has videos on approximating an area by a sum of simple pieces, as a Riemann sum does, and on computing such sums in Python:

- [The Method of Exhaustion](#)
- [Numeric Integration with Python](#)

Before stating the Fundamental Theorem of Calculus, we record two prerequisite results. The usual statement of the Fundamental Theorem of Calculus assumes that the integrand f is continuous on $[a, b]$; continuity is [sufficient](#) there, though not necessary. The two results are “differentiability implies continuity”, which says where continuity comes from, and “continuity implies integrability”, which says what continuity gives us.

Theorem 0.2 (Differentiability implies continuity). *If f is differentiable at $x = c$, then f is continuous at $x = c$.
[2], Theorem 2.1, p. 106*

Proof

Proof. Because $f'(c)$ exists, $f(c)$ is defined, and:

$$\begin{aligned} \lim_{h \rightarrow 0} (f(c+h) - f(c)) &= \lim_{h \rightarrow 0} \left(\frac{f(c+h) - f(c)}{h} \cdot h \right) && \text{(multiply and divide by } h \neq 0\text{)} \\ &= \left(\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \right) \cdot \left(\lim_{h \rightarrow 0} h \right) && \text{(limit of a product, both limits exist)} \\ &= f'(c) \cdot 0 && \text{(definition of } f'(c)\text{)} \\ &= 0 && \text{(multiply)} \end{aligned}$$

So $\lim_{h \rightarrow 0} f(c+h) = f(c)$, which is $\lim_{x \rightarrow c} f(x) = f(c)$ with $x = c+h$; all three conditions of Definition [0.4](#) hold. □

Example 0.15 (Differentiable, hence continuous: $x^3 - x$). $f(x) = x^3 - x$ is differentiable everywhere (with derivative $f'(x) = 3x^2 - 1$), so by Theorem 0.2 it is continuous everywhere.

Example 0.16 (Continuous but not differentiable: $|x|$). The absolute-value function $f(x) = |x|$ is continuous at $x = 0$, since

$$\begin{aligned}\lim_{x \rightarrow 0} |x| &= 0 \\ &= |0|,\end{aligned}$$

but it is not differentiable at $x = 0$: its left-hand derivative there is -1 and its right-hand derivative is $+1$ (Example 6 in Derivatives and Taylor Series).

This counterexample shows that the converse of Theorem 0.2 fails: continuity does not imply differentiability. See Figure 2.

```
ggplot2::ggplot() +
  ggplot2::geom_function(fun = abs, xlim = c(-2, 2), linewidth = 1) +
  ggplot2::geom_point(ggplot2::aes(x = 0, y = 0), size = 3) +
  ggplot2::labs(x = "x", y = expression(f(x) == abs(x))) +
  ggplot2::theme_minimal()
```

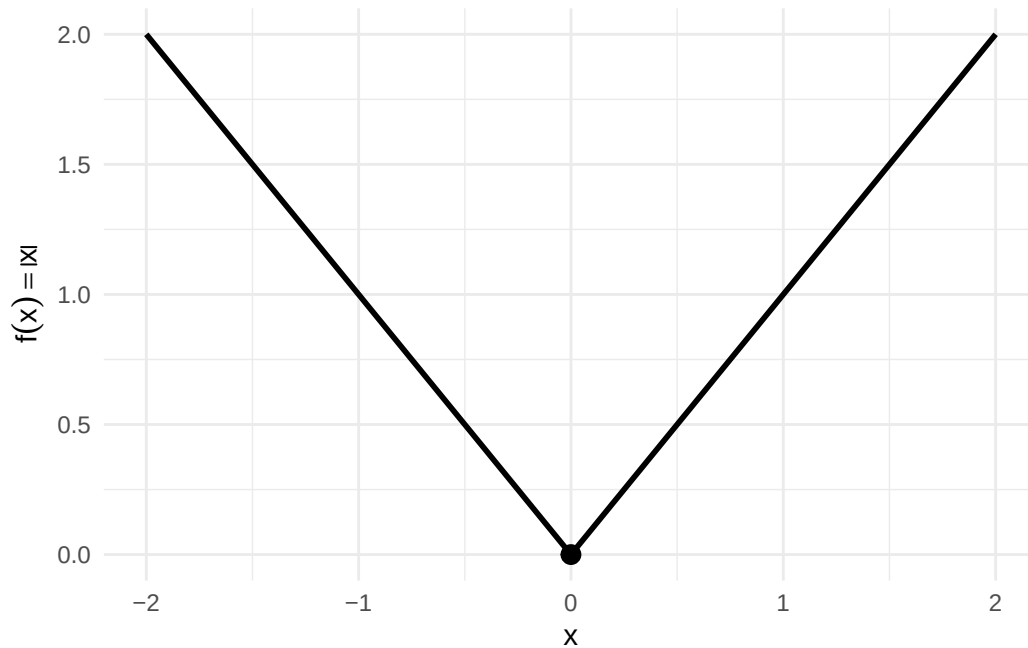


Figure 2: $f(x) = |x|$ has a sharp corner at $x = 0$ (not differentiable there) but is continuous everywhere: no gaps or jumps.

Theorem 0.3 (Continuity implies integrability). *If f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$ (i.e., the Riemann integral $\int_a^b f(x) dx$ exists and is finite).*
[2], Theorem 4.4, p. 272

Example 0.17 (Continuous, hence integrable: polynomials). Every **polynomial** is continuous on $\mathbb{R}$, so by Theorem 0.3 every polynomial is integrable on every closed interval $[a, b]$.

Example 0.18 (Integrable but not continuous: a step function). Let $f(x) = 0$ for $x < \frac{1}{2}$ and $f(x) = 1$ for $x \geq \frac{1}{2}$. Then f has a jump discontinuity (Definition 0.6) at $x = \frac{1}{2}$, but it is integrable on $[0, 1]$:

$$\begin{aligned}\int_0^1 f(x) \, dx &= \int_0^{1/2} 0 \, dx + \int_{1/2}^1 1 \, dx \\ &= 0 + \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

This counterexample shows that the converse of Theorem 0.3 fails: integrability does not imply continuity. See Figure 3.

```

step_df <- tibble::tibble(
  x = c(0, 0.5, 0.5, 1),
  y = c(0, 0, 1, 1),
  segment = c("left", "left", "right", "right")
)

ggplot2::ggplot() +
  ggplot2::geom_rect(
    ggplot2::aes(xmin = 0.5, xmax = 1, ymin = 0, ymax = 1),
    fill = "steelblue", alpha = 0.3
  ) +
  ggplot2::geom_line(
    data = step_df,
    ggplot2::aes(x = x, y = y, group = segment),
    linewidth = 1
  ) +
  ggplot2::geom_point(ggplot2::aes(x = 0.5, y = 0), shape = 1, size = 3) +
  ggplot2::geom_point(ggplot2::aes(x = 0.5, y = 1), shape = 16, size = 3) +
  ggplot2::scale_x_continuous(
    breaks = c(0, 0.5, 1),
    labels = c("0", "1/2", "1")
  ) +
  ggplot2::scale_y_continuous(limits = c(-0.1, 1.2)) +
  ggplot2::labs(x = "x", y = "f(x)") +
  ggplot2::theme_minimal()

```

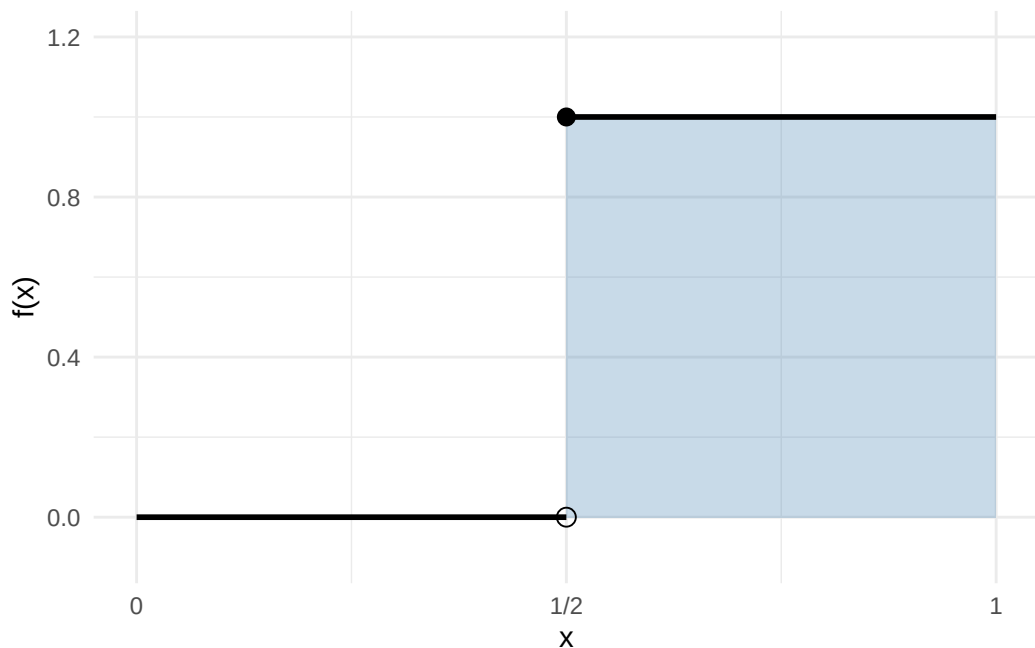


Figure 3: Step function: $f(x) = 0$ on $[0, \frac{1}{2})$ (open circle at the jump) and $f(x) = 1$ on $[\frac{1}{2}, 1]$ (filled circle). The shaded rectangle has area $\frac{1}{2}$, matching the integral computed in Example 0.18.

Together, Theorem 0.2 and Theorem 0.3 establish the chain:

$$\text{differentiable on } [a, b] \Rightarrow \text{continuous on } [a, b] \Rightarrow \text{integrable on } [a, b]$$

Example 0.16 and Example 0.18 show that neither implication reverses in general.

Theorem 0.4 (Equal-width Riemann sums converge to the integral). *If f is Riemann integrable on $[a, b]$ (Definition 0.13), then for every choice of the sample points x_i^* , the equal-width Riemann sums (Definition 0.15) converge to the integral:*

$$\lim_{n \rightarrow \infty} S_n = \int_a^b f(x) dx.$$

i Proof

Proof. The n equal-width subintervals form a partition of $[a, b]$ whose mesh (Definition 0.9) is $(b - a)/n$, which goes to 0 as $n \rightarrow \infty$. So S_n is one of the sums in the limit that defines the integral (Definition 0.12), along a sequence of partitions whose mesh goes to 0, and a limit that has the same value for every choice of partitions has that value along this sequence too. $\square$

Example 0.19 (Equal-width sums for $\int_0^1 x dx$). Let $f(x) = x$ on $[0, 1]$, which is continuous and so Riemann integrable (Theorem 0.3), and take each sample point at the right end of its subinterval, $x_i^* = i/n$. With $\Delta x = 1/n$:

$$\begin{aligned} S_n &= \sum_{i=1}^n \frac{i}{n} \cdot \frac{1}{n} && \text{(equal-width Riemann sum with } x_i^* = i/n) \\ &= \frac{1}{n^2} \sum_{i=1}^n i && \text{(factor out } 1/n^2) \\ &= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} && \text{(sum of the first } n \text{ integers)} \\ &= \frac{n+1}{2n} && \text{(cancel one factor of } n) \end{aligned}$$

So $S_{10} = 0.55$, $S_{100} = 0.505$, $S_{1000} = 0.5005$, and $S_n \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$. By Theorem 0.4, $\int_0^1 x dx = \frac{1}{2}$.

Fundamental Theorem of Calculus

Definition 0.16 (Accumulation function). Let f be Riemann integrable on $[a, b]$ (Definition 0.13). The **accumulation function** of f from a is the function F on $[a, b]$ with $F(a) \stackrel{\text{def}}{=} 0$ and

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t) dt \quad \text{for } a < x \leq b.$$

So $F(x)$ is the integral of f accumulated from a up to x .

Example 0.20 (The accumulation function of a constant). Let $f(t) = 2$ on $[0, 3]$, which is Riemann integrable (Example 0.12). For $0 < x \leq 3$, the computation of Example 0.12, with the interval $[0, 3]$ replaced by $[0, x]$, gives

$$\begin{aligned} F(x) &= \int_0^x 2 dt && \text{(definition of the accumulation function)} \\ &= 2 \cdot (x - 0) && \text{(every Riemann sum is 2 times the total width } x - 0) \\ &= 2x && \text{(subtract)} \end{aligned}$$

and

$$\begin{aligned} F(0) &= 0 \\ &= 2 \cdot 0 \end{aligned}$$

too. For example,

$$\begin{aligned} F(1.5) &= 2 \cdot 1.5 \\ &= 3, \end{aligned}$$

the area of a rectangle of height 2 and width 1.5.

i Remark (Why the integration variable is called t)

Remark 0.2 (Why the integration variable is called t). The letter t inside the integral is a placeholder, renamed from x so that x can serve as the [upper limit](#).

Theorem 0.5 (Fundamental Theorem of Calculus, Part 1 (derivative of an integral)). *Let f be a continuous function on a closed interval $[a, b]$. Let $F(x) = \int_a^x f(t) dt$ for $x \in [a, b]$ be the accumulation function of f from a (Definition 0.16). Then F is differentiable and:*

$$\frac{\partial}{\partial x} \int_a^x f(t) dt = f(x) \quad (1)$$

Note

Continuity on all of $[a, b]$ is a sufficient condition. More generally, Part 1 holds at any individual point x where f is integrable on $[a, b]$ (see Definition 0.13) and continuous at x (see Definition 0.4), even if f has jump discontinuities (Definition 0.6) elsewhere [8], Theorem 6.20, p. 133. This result is the first part of the fundamental theorem of calculus [9].

[2], Theorem 4.11, p. 288

Theorem 0.6 (Fundamental Theorem of Calculus, Part 2 (evaluation theorem)). *Let f be a continuous function on a closed interval $[a, b]$. The F here may be any antiderivative of f — not just the accumulation function from Theorem 0.5. If F is an antiderivative of f on $[a, b]$ (i.e., $\frac{\partial}{\partial x} F(x) = f(x)$ for all $x \in [a, b]$), then:*

$$\int_a^b f(x) dx = F(b) - F(a) \quad (2)$$

Equivalently, with b replaced by a variable upper limit x , integrating the derivative of F recovers the net change in F :

$$\int_a^x F'(t) dt = F(x) - F(a) \quad (3)$$

or equivalently in Leibniz notation:

$$\int_a^x \frac{dF}{dt} dt = F(x) - F(a)$$

[10, Ch. 18], [2], Theorem 4.9, p. 282

The two parts of the FTC together express that **differentiation and integration are inverse operations**:

- Part 1: differentiating the integral of f recovers f (Equation 1).
- Part 2: the integral of f over $[a, b]$ equals the difference of any antiderivative's values at the endpoints (Equation 2), which rearranges to “integrating the derivative of F recovers the net change in F ” (Equation 3).

The standard form of the FTC assumes f is continuous on $[a, b]$; continuity is *sufficient* but not strictly necessary (see the callout note inside Theorem 0.5 for the more general statement). Since differentiability implies continuity (Theorem 0.2), the FTC applies in particular whenever

f is differentiable — a common situation in applied statistics.

Definition 0.17 (Evaluation bracket). For a function F and numbers a and b where F is defined, the **evaluation bracket** is the difference

$$[F(t)]_{t=a}^{t=b} \stackrel{\text{def}}{=} F(b) - F(a).$$

When the variable is clear from the context, it is written $[F(t)]_a^b$. Computing $F(b) - F(a)$ from the bracket is called **evaluating at the limits** (the [limits of integration](#) a and b).

Example 0.21 (Evaluating $\int_1^3 2t \, dt$ with a bracket). $F(t) = t^2$ is an antiderivative of $f(t) = 2t$, since $\frac{\partial}{\partial t} t^2 = 2t$, so

$$\begin{aligned} \int_1^3 2t \, dt &= [t^2]_{t=1}^{t=3} && \text{(FTC Part 2)} \\ &= 3^2 - 1^2 && \text{(definition of the evaluation bracket)} \\ &= 9 - 1 && \text{(square)} \\ &= 8 && \text{(subtract)} \end{aligned}$$

i Remark (The evaluation bracket and the Fundamental Theorem of Calculus)

Remark 0.3 (The evaluation bracket and the Fundamental Theorem of Calculus). With FTC Part 2 (Theorem [0.6](#)), $\int_a^b f(t) \, dt = [F(t)]_{t=a}^{t=b}$ for any antiderivative F of a continuous f .

Exercise 0.1 (Definite integral of a polynomial). Adapted from Miller [11], Question 1.1.31.

Evaluate the integral:

$$\int_0^1 (x^4 + x^2 + 1) \, dx$$

Solution

Solution 0.1. Integrate term by term using the power rule for integration:

$$\begin{aligned}
\int_0^1 (x^4 + x^2 + 1) dx &= \left[\frac{x^5}{5} + \frac{x^3}{3} + x \right]_0^1 \\
&= \left(\frac{1}{5} + \frac{1}{3} + 1 \right) - 0 \\
&= \frac{3 + 5 + 15}{15} \\
&= \frac{23}{15}
\end{aligned}$$

Exercise 0.2 (Integral of a perfect square). Adapted from Miller [11], Question 1.1.32. Evaluate the integral:

$$\int_0^1 (x^2 + 2x + 1) dx$$

Solution

Solution 0.2. Notice that the integrand is a perfect square: $x^2 + 2x + 1 = (x + 1)^2$. With substitution $u = x + 1$ (where $du = dx$):

$$\begin{aligned}
\int_0^1 (x + 1)^2 dx &= \left[\frac{(x + 1)^3}{3} \right]_0^1 \\
&= \frac{2^3}{3} - \frac{1^3}{3} \\
&= \frac{8 - 1}{3} \\
&= \frac{7}{3}
\end{aligned}$$

Alternative (term by term):

$$\begin{aligned}
\left[\frac{x^3}{3} + x^2 + x \right]_0^1 &= \frac{1}{3} + 1 + 1 \\
&= \frac{7}{3}
\end{aligned}$$

Exercise 0.3 (Integral of a composite power). Adapted from Miller [11], Question 1.1.33. Evaluate the integral:

$$\int_0^1 (x^2 + 2x + 1)^2 dx$$

Solution

Solution 0.3. Recognize that $x^2 + 2x + 1 = (x + 1)^2$, so the integrand is:

$$\begin{aligned}(x^2 + 2x + 1)^2 &= [(x + 1)^2]^2 \\ &= (x + 1)^4\end{aligned}$$

Using the substitution $u = x + 1$ with $du = dx$:

$$\begin{aligned}\int_0^1 (x + 1)^4 dx &= \left[\frac{(x + 1)^5}{5} \right]_0^1 \\ &= \frac{2^5 - 1^5}{5} \\ &= \frac{32 - 1}{5} \\ &= \frac{31}{5}\end{aligned}$$

Remark: Do not mistakenly write $\int (x^2 + 2x + 1)^2 dx = \frac{(x^2 + 2x + 1)^3}{3}$; that formula requires the derivative of the inside function $(2x + 2)$ to be present as a factor in the integrand.

Exercise 0.4 (Integral of odd trigonometric functions). Adapted from Miller [11], Question 1.1.34.

Evaluate the integral:

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x) dx$$

Solution

Solution 0.4. Because $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$:

$$\sin^3(-x) \cos(-x) + \sin(-x) \cos(-x) = -\sin^3 x \cos x - \sin x \cos x$$

The integrand is an odd function. The integral of any continuous odd function over a symmetric interval $[-a, a]$ is zero:

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x) dx = 0$$

Alternative (by substitution): Set $u = \sin x$, so $du = \cos x dx$. As x ranges from $-\pi/2$ to $\pi/2$, u ranges from -1 to 1 :

$$\begin{aligned} \int_{-1}^1 (u^3 + u) du &= \left[\frac{u^4}{4} + \frac{u^2}{2} \right]_{-1}^1 \\ &= \left(\frac{1}{4} + \frac{1}{2} \right) - \left(\frac{1}{4} + \frac{1}{2} \right) \\ &= 0 \end{aligned}$$

Exercise 0.5 (Separating even and odd parts of a polynomial). Adapted from Miller [11], Question 1.1.35.

Evaluate the integral:

$$\int_{-4}^4 (x^3 + 6x^2 - 2x - 3) dx$$

Solution

Solution 0.5. Split the integrand into its odd part $x^3 - 2x$ and its even part $6x^2 - 3$:

$$\int_{-4}^4 (x^3 - 2x) dx = 0$$

The even part doubles over the half-interval $[0, 4]$:

$$\begin{aligned} \int_{-4}^4 (6x^2 - 3) dx &= 2 \int_0^4 (6x^2 - 3) dx \\ &= 2[2x^3 - 3x]_0^4 \\ &= 2[2(64) - 3(4)] \\ &= 2(128 - 12) \\ &= 232 \end{aligned}$$

Exercise 0.6 (Substitution leading to a natural logarithm). Adapted from Miller [11], Question 1.1.36.

Evaluate the integral:

$$\int_0^1 \frac{x}{1+x^2} dx$$

Solution

Solution 0.6. Let $u = 1 + x^2$. Then $du = 2x dx$, so $x dx = \frac{1}{2} du$. Transform the limits of integration:

- When $x = 0$:

$$\begin{aligned} u &= 1 + 0^2 \\ &= 1. \end{aligned}$$

- When $x = 1$:

$$\begin{aligned} u &= 1 + 1^2 \\ &= 2. \end{aligned}$$

Applying the substitution:

$$\begin{aligned} \int_0^1 \frac{x}{1+x^2} dx &= \frac{1}{2} \int_1^2 \frac{du}{u} \\ &= \frac{1}{2} [\log u]_1^2 \\ &= \frac{1}{2} (\log 2 - \log 1) \\ &= \frac{\log 2}{2} \end{aligned}$$

Remark: When using u -substitution with definite integrals, the transformation $x \mapsto u(x)$ must be one-to-one on the domain of integration.

Exercise 0.7 (Substitution with a polynomial power). Adapted from Miller [11], Question 1.1.37.

Evaluate the integral:

$$\int_0^3 (x^3 + 3x)^8 (x^2 + 1) dx$$

Solution

Solution 0.7. Let $u = x^3 + 3x$. Then:

$$\begin{aligned} du &= (3x^2 + 3) dx \\ &= 3(x^2 + 1) dx \\ \Rightarrow (x^2 + 1) dx &= \frac{du}{3} \end{aligned}$$

Limits:

- When $x = 0$: $u = 0$.
- When $x = 3$:

$$\begin{aligned} u &= 3^3 + 3(3) \\ &= 27 + 9 \\ &= 36. \end{aligned}$$

Substitute:

$$\begin{aligned} \int_0^3 (x^3 + 3x)^8 (x^2 + 1) dx &= \frac{1}{3} \int_0^{36} u^8 du \\ &= \frac{1}{3} \left[\frac{u^9}{9} \right]_0^{36} \\ &= \frac{36^9}{27} \end{aligned}$$

Exercise 0.8 (Substitution with a trigonometric integrand). Adapted from Miller [11], Question 1.1.38.

Evaluate the integral:

$$\int_0^2 x \cos(3x^2) dx$$

Solution

Solution 0.8. Let $u = 3x^2$. Then $du = 6x dx$, meaning $x dx = \frac{du}{6}$.

Limits:

- When $x = 0$: $u = 0$.

- When $x = 2$:

$$\begin{aligned} u &= 3(2^2) \\ &= 12. \end{aligned}$$

Substitute:

$$\begin{aligned} \int_0^2 x \cos(3x^2) dx &= \frac{1}{6} \int_0^{12} \cos u du \\ &= \frac{1}{6} [\sin u]_0^{12} \\ &= \frac{\sin 12}{6} \end{aligned}$$

Exercise 0.9 (Integration by parts with an exponential factor). Adapted from Miller [11], Question 1.1.40.

Evaluate the integral:

$$\int_a^b x^3 \exp\left\{-\frac{x^2}{2}\right\} dx$$

Solution

Solution 0.9. Factor the integrand as $x^2 \cdot (x \exp\{-x^2/2\})$. Set:

$$u = x^2, \quad dv = x \exp\left\{-\frac{x^2}{2}\right\} dx$$

Then $du = 2x dx$ and $v = -\exp\{-x^2/2\}$.

Using integration by parts ($\int u dv = uv - \int v du$):

$$\begin{aligned}
\int_a^b x^3 \exp\left\{-\frac{x^2}{2}\right\} dx &= \left[-x^2 \exp\left\{-\frac{x^2}{2}\right\}\right]_a^b - \int_a^b \left[-\exp\left\{-\frac{x^2}{2}\right\}\right] (2x dx) \\
&= \left[-x^2 \exp\left\{-\frac{x^2}{2}\right\}\right]_a^b + 2 \int_a^b x \exp\left\{-\frac{x^2}{2}\right\} dx \\
&= \left[-x^2 \exp\left\{-\frac{x^2}{2}\right\}\right]_a^b + 2 \left[-\exp\left\{-\frac{x^2}{2}\right\}\right]_a^b \\
&= \left[-(x^2 + 2) \exp\left\{-\frac{x^2}{2}\right\}\right]_a^b \\
&= (a^2 + 2) \exp\left\{-\frac{a^2}{2}\right\} - (b^2 + 2) \exp\left\{-\frac{b^2}{2}\right\}
\end{aligned}$$

Example 0.22 (FTC Part 1 visualized: accumulation function for $f(t) = 2t$). Take $f(t) = 2t$ on $[0, 2]$. The accumulation function from 0 is

$$\begin{aligned}
F(x) &\stackrel{\text{def}}{=} \int_0^x 2t dt \\
&= [t^2]_{t=0}^{t=x} \\
&= x^2 - 0^2 \\
&= x^2,
\end{aligned}$$

so $F(x) = x^2$, and indeed

$$\begin{aligned}
F'(x) &= 2x \\
&= f(x),
\end{aligned}$$

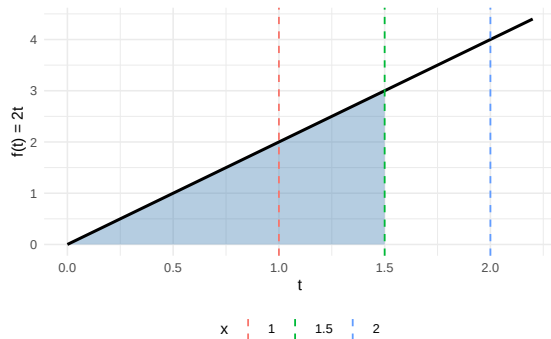
as Theorem 0.5 predicts. Figure 4 shows the integrand on the left (shaded area equals $F(x)$ at each x) and the accumulation function $F(x) = x^2$ on the right (its slope at x equals $f(x) = 2x$).

```

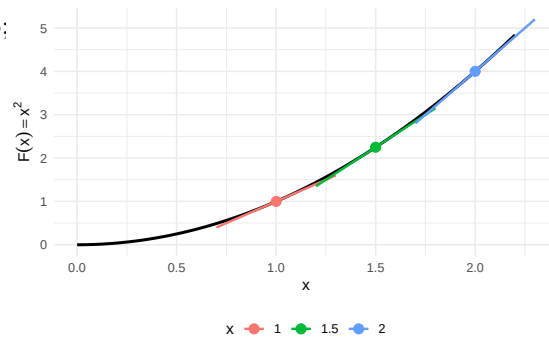
ggplot2::ggplot() +
  ggplot2::geom_area(
    data = tibble::tibble(t = seq(0,
      ggplot2::aes(x = t, y = 2 * t),
      fill = "steelblue", alpha = 0.4 )
    ) +
    ggplot2::geom_function(fun = \(t)
    ggplot2::geom_vline(
      data = tibble::tibble(x = x_marks,
        ggplot2::aes(xintercept = x, col
        linetype = "dashed", linewidth =
      ) +
      ggplot2::labs(x = "t", y = "f(t) =
      ggplot2::theme_minimal() +
      ggplot2::theme(legend.position = "

slope_df <- tibble::tibble(
  x = x_marks,
  Fx = x_marks^2,
  slope = 2 * x_marks
) +
ggplot2::ggplot() +
  ggplot2::geom_function(fun = \(x) x^2, xlim = c(0, 2
  ggplot2::geom_point(
    data = slope_df,
    ggplot2::aes(x = x, y = Fx, color = factor(x)),
    size = 3
  ) +
  ggplot2::geom_segment(
    data = slope_df,
    ggplot2::aes(
      x = x - 0.3, xend = x + 0.3,
      y = Fx - 0.3 * slope, yend = Fx + 0.3 * slope,
      color = factor(x)
    ),
    linewidth = 0.8
  ) +
  ggplot2::labs(x = "x", y = expression(F(x) == x^2),
  ggplot2::theme_minimal() +
  ggplot2::theme(legend.position = "bottom")

```



(a) $f(t) = 2t$; shaded area equals $F(1.5) = 2.25$; vertical lines mark $x \in \{1, 1.5, 2\}$.



(b) $F(x) = x^2$; tangent slope at each marked x equals $f(x) = 2x$.

Left: $f(t) = 2t$; the shaded area

$$\int_0^{1.5} 2t \, dt = F(1.5) = 2.25;$$

Figure 4: vertical lines mark $x \in \{1, 1.5, 2\}$. Right: $F(x) = x^2$; for each marked x , the tangent slope equals $f(x) = 2x$.

Example 0.23 (CDF and PDF of the exponential distribution). In what follows, f denotes the PDF and F the CDF — the same letters as the antiderivative pair in Definition 0.1, because the FTC will show F is exactly an antiderivative of f .

Let T be a [random variable](#) with the [exponential distribution](#) with rate $\lambda > 0$. Its [probability density function \(PDF\)](#) is [12, Sec. II], p. 295, “Survival and Hazard Functions for Selected Distributions”:

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

FTC Part 2 gives the [cumulative distribution function \(CDF\)](#), $F(t) = P(T \leq t)$, from the PDF. Apply the e^{cx} rule from Theorem 0.1 with $c = -\lambda$ to antidifferentiate the integrand:

$$\begin{aligned} F(t) &= \int_0^t \lambda e^{-\lambda u} du && \text{(the CDF integrates the PDF)} \\ &= \left[\lambda \cdot \frac{1}{-\lambda} e^{-\lambda u} \right]_{u=0}^{u=t} && \text{(FTC Part 2, with the } e^{cx} \text{ rule)} \\ &= [(-1)e^{-\lambda u}]_{u=0}^{u=t} && (\lambda/(-\lambda) = -1) \\ &= [-e^{-\lambda u}]_{u=0}^{u=t} && \text{(multiply by } -1) \\ &= -e^{-\lambda t} - (-e^0) && \text{(evaluate at the limits)} \\ &= -e^{-\lambda t} - (-1) && (e^0 = 1) \\ &= 1 - e^{-\lambda t} && \text{(rearrange)} \end{aligned}$$

FTC Part 1 recovers the PDF from the CDF:

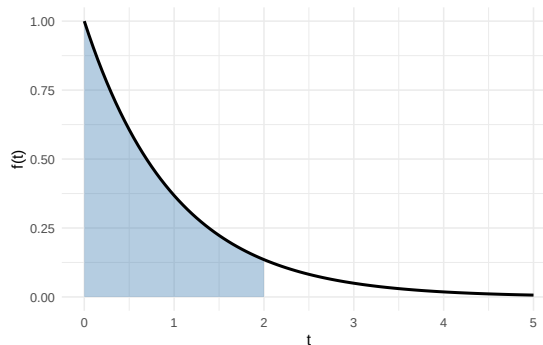
$$\begin{aligned} \frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} (1 - e^{-\lambda t}) && \text{(substitute } F) \\ &= \frac{\partial}{\partial t} 1 - \frac{\partial}{\partial t} e^{-\lambda t} && \text{(derivative of a difference)} \\ &= 0 - \frac{\partial}{\partial t} e^{-\lambda t} && \text{(constant rule)} \\ &= 0 - e^{-\lambda t} \cdot \frac{\partial}{\partial t} (-\lambda t) && \text{(chain rule, with inner function } -\lambda t) \\ &= 0 - e^{-\lambda t} \cdot (-\lambda) && \text{(constant multiple rule)} \\ &= \lambda e^{-\lambda t} && \text{(simplify)} \\ &= f(t) && \text{(definition of } f) \end{aligned}$$

For a concrete instance: with $\lambda = 1$, the probability that $T \leq 2$ is:

$$\begin{aligned} F(2) &= 1 - e^{-1.2} \\ &= 1 - e^{-2} \\ &\approx 1 - 0.135 \\ &= 0.865 \end{aligned}$$

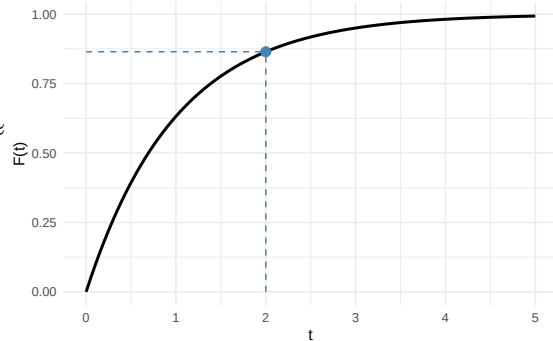
See Figure [5](#).

```
ggplot2::ggplot() +
  ggplot2::geom_area(
    data = tibble::tibble(t = seq(0,
      ggplot2::aes(x = t, y = lambda *
        fill = "steelblue", alpha = 0.4
    ) +
  ggplot2::geom_function(
    fun = \(t) lambda * exp(-lambda * t),
    xlim = c(0, t_max), linewidth = 1
  ) +
  ggplot2::labs(x = "t", y = "f(t)")
  ggplot2::theme_minimal()
```



(a) PDF with $\lambda = 1$; shaded area equals $F(2) \approx 0.865$.

```
ggplot2::ggplot() +
  ggplot2::geom_function(
    fun = \(t) 1 - exp(-lambda * t),
    xlim = c(0, t_max), linewidth = 1
  ) +
  ggplot2::geom_point(
    ggplot2::aes(x = t_focus, y = F_at_focus),
    size = 3, color = "steelblue"
  ) +
  ggplot2::geom_segment(
    ggplot2::aes(x = t_focus, xend = t_focus, y = 0, yend = F_at_focus),
    linetype = "dashed", color = "steelblue"
  ) +
  ggplot2::geom_segment(
    ggplot2::aes(x = 0, xend = t_focus, y = F_at_focus, yend = F_at_focus),
    linetype = "dashed", color = "steelblue"
  ) +
  ggplot2::labs(x = "t", y = "F(t)") +
  ggplot2::theme_minimal()
```



(b) CDF with $\lambda = 1$; point marks $F(2) \approx 0.865$.

Figure 5: Exponential distribution with $\lambda = 1$. Left: the PDF $f(t) = \lambda e^{-\lambda t}$; the shaded area under the curve from 0 to 2 equals $F(2) \approx 0.865$. Right: the CDF $F(t) = 1 - e^{-\lambda t}$; the dashed lines mark the value $F(2)$ computed via FTC Part 2.

Exercise 0.10 (Improper Gaussian-kernel integral). Adapted from Miller [11], Question 1.1.39.

Evaluate the improper integral:

$$\int_0^\infty x \exp\left\{-\frac{x^2}{4}\right\} dx$$

Solution

Solution 0.10. Let $u = x^2/4$. Then $du = \frac{x}{2} dx$, so $x dx = 2 du$.

Limits:

- As $x \rightarrow 0$: $u \rightarrow 0$.
- As $x \rightarrow \infty$: $u \rightarrow \infty$.

Substitute:

$$\begin{aligned} \int_0^\infty x \exp\left\{-\frac{x^2}{4}\right\} dx &= 2 \int_0^\infty e^{-u} du \\ &= 2[-e^{-u}]_0^\infty \\ &= 2(0 - (-1)) \\ &= 2 \end{aligned}$$

Remark: Integrals of this structure arise directly in calculating moments and normalization constants for Gaussian and Rayleigh probability distributions.

Exercise 0.11 (Convolution of two uniform densities). Adapted from Miller [11], Question 1.1.41.

Let $f(x) = \mathbb{1}_{[0,1]}(x)$ be the indicator of the unit interval:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Evaluate the convolution:

$$(f * f)(x) = \int_{-\infty}^\infty f(t)f(x-t) dt$$

for all $x \in \mathbb{R}$.

Solution

Solution 0.11. The integrand $f(t)f(x-t)$ equals 1 when both $t \in [0, 1]$ and $x-t \in [0, 1]$, and equals 0 otherwise.

The condition $0 \leq x-t \leq 1$ rearranges to:

$$x - 1 \leq t \leq x$$

For this interval to overlap with $t \in [0, 1]$, we must have $x \geq 0$ and $x - 1 \leq 1$, meaning $0 \leq x \leq 2$. Outside $[0, 2]$, $(f * f)(x) = 0$.

For $x \in [0, 2]$, we distinguish two cases:

- **Case 1:** $0 \leq x \leq 1$. The overlapping bounds for t are $[0, x]$:

$$\begin{aligned}(f * f)(x) &= \int_0^x 1 \, dt \\ &= x\end{aligned}$$

- **Case 2:** $1 \leq x \leq 2$. The overlapping bounds for t are $[x - 1, 1]$:

$$\begin{aligned}(f * f)(x) &= \int_{x-1}^1 1 \, dt \\ &= 1 - (x - 1) \\ &= 2 - x\end{aligned}$$

Combining the cases:

$$(f * f)(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Remark: In probability theory, if $X_1, X_2 \sim \text{Uniform}(0, 1)$ independently, the probability density function of their sum $S = X_1 + X_2$ is the convolution of their individual densities, producing this symmetric triangular distribution on $[0, 2]$.

Theorem 0.7 (Convolution of two uniform densities). *Let $f(x) = \mathbb{1}_{[0,1]}(x)$ be the indicator of the unit interval. Then for every $x \in \mathbb{R}$,*

$$(f * f)(x) = \int_{-\infty}^{\infty} f(t)f(x - t) \, dt = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof

Proof. This is the solution to Exercise 0.11, worked in Solution 0.11. The integrand $f(t)f(x-t)$ equals 1 when both $t \in [0, 1]$ and $x-t \in [0, 1]$, and 0 otherwise. The second condition is $x-1 \leq t \leq x$, so the integrand is 1 exactly on the overlap $[\max(0, x-1), \min(1, x)]$, and the integral is the length of that overlap.

- If $x < 0$ or $x > 2$, the overlap is empty, so $(f * f)(x) = 0$.
- If $0 \leq x \leq 1$, the overlap is $[0, x]$, so $(f * f)(x) = \int_0^x 1 \, dt = x$.
- If $1 < x \leq 2$, the overlap is $[x-1, 1]$, so $(f * f)(x) = \int_{x-1}^1 1 \, dt = 2 - x$.

□

Videos

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist has videos on evaluating definite integrals:

- [Definite Integrals](#)
- [Definite Integral Exercise](#)

- [1] Wikipedia contributors, “Antiderivative — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Antiderivative>
- [2] R. Larson and B. H. Edwards, *Calculus*, 11th ed. Boston, MA: Cengage Learning, 2018. Available: <https://www.cengage.com/c/calculus-11e-larson/>
- [3] Wikipedia contributors, “Continuous function — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Continuous_function
- [4] Wikipedia contributors, “Classification of discontinuities — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Classification_of_discontinuities
- [5] Wikipedia contributors, “Partition of an interval — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Partition_of_an_interval
- [6] Wikipedia contributors, “Riemann sum — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Riemann_sum
- [7] Wikipedia contributors, “Riemann integral — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Riemann_integral
- [8] W. Rudin, *Principles of mathematical analysis*, 3rd ed. in International series in pure and applied mathematics. New York: McGraw-Hill, 1976.
- [9] Wikipedia contributors, “Fundamental theorem of calculus — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus

- [10] A. D. Banner, *The calculus lifesaver : All the tools you need to excel at calculus*. in A princeton lifesaver study guide. Princeton, New Jersey: Princeton University Press, 2007. Available: <https://press.princeton.edu/books/paperback/9780691130880/the-calculus-lifesaver>
- [11] S. J. Miller, *The probability lifesaver: Calculus review problems*. 2016. Available: https://web.williams.edu/Mathematics/sjmiller/public_html/probabilitylifesaver/index.htm#:~:text=http%3A//web.williams.edu/Mathematics/sjmiller/public_html/probabilitylifesaver/supplementalchap_calcreview.pdf
- [12] D. G. Kleinbaum and M. Klein, *Survival analysis: A self-learning text*, 3rd ed. Springer, 2012. doi: [10.1007/978-1-4419-6646-9](https://doi.org/10.1007/978-1-4419-6646-9).