

Double Integrals and Fubini's Theorem

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Double Integrals

The **Fubini–Tonelli theorem** states conditions under which the order of integration in a double integral can be exchanged [1]. We state them in two settings: the Riemann version (Theorem 0.1) is what applied courses usually use for double integrals of continuous functions on simple regions; the σ -finite measure-theoretic versions (Theorem 0.4 and Theorem 0.5) are included to make the [joint-distribution form](#) corollary in *Probability for Data Science* follow from a stated theorem rather than from an aside.

Definition 0.1 (Double integral). Let f be a bounded function on a closed, bounded plane region $R \subseteq \mathbb{R}^2$ [2]. Cover R with a grid of rectangles, keep the n rectangles that lie entirely inside R , with areas $\Delta A_1, \dots, \Delta A_n$, and choose a point (x_i, y_i) in the i -th rectangle. The **double integral** of f over R is

$$\iint_R f(x, y) dA \stackrel{\text{def}}{=} \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i) \Delta A_i,$$

where $\|\Delta\|$ is the length of the longest diagonal among the n rectangles, when that limit exists and has the same value for every choice of grids and of the points (x_i, y_i) . The symbol dA stands for an element of area.
[3, Sec. 14.2]

Example 0.1 (The double integral of 1 is an area). Let $f(x, y) = 1$ on the rectangle $R = [0, 2] \times [0, 3]$. Every sum in Definition 0.1 adds up the areas of rectangles inside R , and those sums approach the area of R as the grid gets finer, so

$$\begin{aligned} \iint_R 1 dA &= 2 \cdot 3 \\ &= 6. \end{aligned}$$

Definition 0.2 (Iterated integral). An **iterated integral** is an integral of an integral [4]:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \stackrel{\text{def}}{=} \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx$$

That is, integrate over the inner variable (y) first, holding the outer variable (x) fixed, and then integrate the result over the outer variable. The inner limits $g_1(x)$ and $g_2(x)$ may depend on the outer variable, or be constants c and d , as in $\int_a^b \int_c^d f(x, y) dy dx$. Iterated integrals in the other order, $\int \int \cdots dx dy$, are defined the same way with the roles of x and y swapped.

Example 0.2 (An iterated integral). Integrating over y first, then x :

$$\begin{aligned} \int_0^1 \int_0^2 xy dy dx &= \int_0^1 \left(\int_0^2 xy dy \right) dx && \text{(definition of the iterated integral)} \\ &= \int_0^1 x \left(\int_0^2 y dy \right) dx && (x \text{ is constant in } y) \\ &= \int_0^1 x \left[\frac{y^2}{2} \right]_{y=0}^{y=2} dx && \text{(antiderivative of } y) \\ &= \int_0^1 2x dx && \text{(evaluate at the limits)} \\ &= [x^2]_{x=0}^{x=1} && \text{(antiderivative of } 2x) \\ &= 1 && \text{(evaluate at the limits)} \end{aligned}$$

Theorem 0.1 (Fubini's theorem (Riemann version)). Let f be **continuous** on a plane region $R \subseteq \mathbb{R}^2$ [5].

1. **Vertically simple region.** If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

2. **Horizontally simple region.** If R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

When R can be described both ways, the two iterated integrals are equal — so the order of integration can be exchanged.
 [3], Theorem 14.2, p. 982

Example 0.3 (Changing the order of integration for a non-rectangular region). Adapted from [3, Sec. 14.2], Example 4, pp. 984–985.

Let X and Y be independent Uniform(0, 1) random variables, with joint density $f(x, y) = 1$ on the unit square $[0, 1]^2$. Define the function $g(x, y) = e^{-x^2} \mathbb{1}(y \leq x)$, where e is Euler's number, and compute its expectation $\mathbb{E}[g(X, Y)]$.

Because the joint density equals 1 on $[0, 1]^2$, this expectation is the double integral of g over the unit square:

$$\mathbb{E}[g(X, Y)] = \iint_{[0, 1]^2} g(x, y) dA.$$

The indicator factor $\mathbb{1}(y \leq x)$ equals 1 on the triangular region where $y \leq x$ and 0 elsewhere, so only that region, namely $D = \{(x, y) : x \in [0, 1], y \in [0, x]\}$ (Figure 1), contributes, and there $g(x, y) = e^{-x^2}$:

$$\mathbb{E}[g(X, Y)] = \iint_D e^{-x^2} dA.$$

```

region <- tibble::tibble(x = c(0, 1, 1), y = c(0, 0, 1))
ggplot2::ggplot(region, ggplot2::aes(x = x, y = y)) +
  ggplot2::geom_polygon(
    fill = "steelblue", alpha = 0.4, color = "black", linewidth = 0.7
  ) +
  ggplot2::annotate(
    "text", x = 0.65, y = 0.28, label = "D", size = 8, fontface = "italic"
  ) +
  ggplot2::annotate(
    "text", x = 0.36, y = 0.52, label = "y == x",
    parse = TRUE, angle = 45
  ) +
  ggplot2::annotate(
    "text", x = 1.08, y = 0.5, label = "x == 1",
    parse = TRUE, angle = 90, hjust = 0.5
  ) +
  ggplot2::annotate(
    "text", x = 0.5, y = -0.1, label = "y == 0", parse = TRUE
  ) +
  ggplot2::scale_x_continuous(breaks = c(0, 0.5, 1), limits = c(-0.1, 1.3)) +
  ggplot2::scale_y_continuous(breaks = c(0, 0.5, 1), limits = c(-0.2, 1.2)) +
  ggplot2::coord_equal() +
  ggplot2::labs(x = "First coordinate, x", y = "Second coordinate, y") +
  ggplot2::theme_minimal()

```

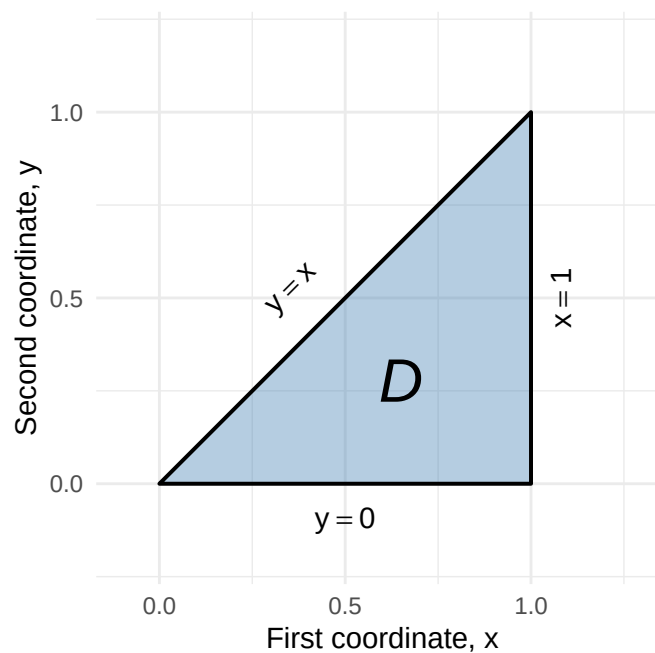


Figure 1: Triangular integration region $D = \{(x, y) : x \in [0, 1], y \in [0, x]\}$, bounded below by $y = 0$, above-left by $y = x$, and on the right by $x = 1$.

Order $dx\,dy$ is intractable. Re-describing D as $D = \{(x, y) : y \in [0, 1], x \in [y, 1]\}$, the inner integral is

$$\int_y^1 e^{-x^2} dx,$$

which cannot be computed with FTC Part 2: no antiderivative of e^{-x^2} can be written as a finite formula built from powers, exponentials, logarithms, and trigonometric functions (an **elementary** antiderivative).

Order $dy\,dx$ works. Applying Theorem 0.1 Part 1 (e^{-x^2} is continuous and D is the vertically simple region $x \in [0, 1], y \in [0, x]$):

$$\begin{aligned} \iint_D e^{-x^2} dA &= \int_0^1 \int_0^x e^{-x^2} dy\,dx && \text{(Fubini, vertically simple region)} \\ &= \int_0^1 e^{-x^2} \left(\int_0^x dy \right) dx && (e^{-x^2} \text{ is constant in } y) \\ &= \int_0^1 x e^{-x^2} dx && (\int_0^x dy = x) \end{aligned}$$

To antidifferentiate $x e^{-x^2}$, let $u = x^2$ be the inner function. By the chain rule (Theorem 10 in Derivatives and Taylor Series), with $\frac{du}{dx} = 2x$,

$$\begin{aligned} \frac{d}{dx} \left(-\frac{1}{2} e^{-u} \right) &= -\frac{1}{2} e^{-u} \cdot (-1) \cdot \frac{du}{dx} && \text{(chain rule)} \\ &= -\frac{1}{2} e^{-x^2} \cdot (-1) \cdot 2x && \text{(substitute } u = x^2 \text{ and } du/dx = 2x) \\ &= x e^{-x^2} && \text{(multiply)} \end{aligned}$$

so $-\frac{1}{2} e^{-x^2}$ is an antiderivative of $x e^{-x^2}$, and

$$\begin{aligned} \int_0^1 x e^{-x^2} dx &= \left[-\frac{1}{2} e^{-x^2} \right]_0^1 && \text{(FTC Part 2)} \\ &= -\frac{1}{2} (e^{-1} - e^0) && \text{(evaluate at the limits)} \\ &= -\frac{1}{2} (e^{-1} - 1) && (e^0 = 1) \\ &= \frac{1 - e^{-1}}{2} && \text{(distribute } -\frac{1}{2}) \\ &= \frac{e - 1}{2e} && \text{(multiply numerator and denominator by } e) \\ &\approx 0.316 \end{aligned}$$

The solid whose volume equals this integral is shown in Figure 2.

(This 3D surface is interactive; see the HTML or slide version of this page to view it.)

Figure 2: Surface $z = e^{-x^2}$ over the region $D = \{(x, y) : x \in [0, 1], y \in [0, x]\}$. The surface depends only on x (constant in y), so for each x the inner integral over $y \in [0, x]$ contributes $x \cdot e^{-x^2}$.

Exercise 0.1 (Double integral over a triangular region). Adapted from Miller [6], Question 1.1.50.

Evaluate the integral:

$$\int_0^1 \int_0^x xy \, dy \, dx$$

and confirm the result by reversing the order of integration.

Solution

Solution 0.1. 1. Given order (y then x):

$$\begin{aligned} \int_0^x xy \, dy &= x \left[\frac{y^2}{2} \right]_0^x \\ &= \frac{x^3}{2} \\ \int_0^1 \frac{x^3}{2} \, dx &= \left[\frac{x^4}{8} \right]_0^1 \\ &= \frac{1}{8} \end{aligned}$$

2. Reversed order (x then y): The triangular region $T = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$ is equivalently described by $T = \{(x, y) : 0 \leq y \leq 1, y \leq x \leq 1\}$:

$$\begin{aligned} \int_0^1 \int_y^1 xy \, dx \, dy &= \int_0^1 y \left[\frac{x^2}{2} \right]_{x=y}^1 dy \\ &= \int_0^1 \frac{y(1-y^2)}{2} dy \\ \int_0^1 \left(\frac{y}{2} - \frac{y^3}{2} \right) dy &= \left[\frac{y^2}{4} - \frac{y^4}{8} \right]_0^1 \\ &= \frac{1}{4} - \frac{1}{8} \\ &= \frac{1}{8} \end{aligned}$$

Both orders yield $1/8$.

Exercise 0.2 (Switching order of integration on a non-rectangular region). Adapted from Miller [6], Question 1.1.51.

Express the integral:

$$\int_0^1 \int_0^x ye^{-xy} dy dx$$

by reversing the order of integration.

Solution

Solution 0.2. In the given order, the inner integral $\int_0^x ye^{-xy} dy$ requires integration by parts with respect to y .

Switching the order over the triangular domain $\{(x, y) : 0 \leq y \leq 1, y \leq x \leq 1\}$:

$$\int_0^1 \int_y^1 ye^{-xy} dx dy$$

Now the inner integral has the factor y in place for direct integration with respect to x :

$$\begin{aligned} \int_y^1 ye^{-xy} dx &= [-e^{-xy}]_{x=y}^1 \\ &= e^{-y^2} - e^{-y} \end{aligned}$$

The double integral becomes:

$$\begin{aligned} \int_0^1 (e^{-y^2} - e^{-y}) dy &= \int_0^1 e^{-y^2} dy - [-e^{-y}]_0^1 \\ &= \int_0^1 e^{-y^2} dy - (1 - e^{-1}) \end{aligned}$$

The term $\int_0^1 e^{-y^2} dy$ has no elementary antiderivative; it can be written using the error function $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ as $\frac{\sqrt{\pi}}{2} \operatorname{erf}(1) \approx 0.7468$. The overall value is:

$$\frac{\sqrt{\pi}}{2} \operatorname{erf}(1) + e^{-1} - 1 \approx 0.1147$$

Example 0.4 (When conditions fail: a counterexample). The conditions in Theorem 0.1 are not merely technical — when they fail, iterated integrals can exist yet disagree.

Let

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

on the unit square $R = [0, 1] \times [0, 1]$. Strictly, f is defined on $R \setminus \{(0, 0)\}$: the denominator vanishes at the origin, so f is undefined there (we return to this point in the condition check).

Integrating y first, then x :

Holding x fixed and differentiating in y (by the quotient rule, [Theorem 9 in Derivatives and Taylor Series](#)),

$$\begin{aligned} \frac{d}{dy} \frac{y}{x^2 + y^2} &= \frac{(x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} \\ &= \frac{x^2 - y^2}{(x^2 + y^2)^2}. \end{aligned}$$

(A derivative in one variable with the others held fixed is a [partial derivative](#), defined on the vector calculus page.) The arctangent $\arctan$ is the inverse of the tangent function from trigonometry; all this example needs is that $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$, $\arctan(0) = 0$, and $\arctan(1) = \frac{\pi}{4}$.

$$\begin{aligned} \int_0^1 \int_0^1 f(x, y) dy dx &= \int_0^1 \left[\frac{y}{x^2 + y^2} \right]_{y=0}^{y=1} dx \\ &= \int_0^1 \frac{1}{x^2 + 1} dx \\ &= [\arctan(x)]_0^1 \\ &= \frac{\pi}{4} \end{aligned}$$

Integrating x first, then y :

Holding y fixed and differentiating in x ,

$$\begin{aligned} \frac{d}{dx} \left(-\frac{x}{x^2 + y^2} \right) &= -\frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} \\ &= \frac{x^2 - y^2}{(x^2 + y^2)^2} : \end{aligned}$$

$$\begin{aligned}
\int_0^1 \int_0^1 f(x, y) \, dx \, dy &= \int_0^1 \left[-\frac{x}{x^2 + y^2} \right]_{x=0}^{x=1} dy \\
&= \int_0^1 \left(-\frac{1}{1 + y^2} \right) dy \\
&= -[\arctan(y)]_0^1 \\
&= -\frac{\pi}{4}
\end{aligned}$$

Conclusion: $\frac{\pi}{4} \neq -\frac{\pi}{4}$, so the two iterated integrals are unequal. Theorem 0.1 does not apply here.

Why Theorem 0.1's condition fails: Theorem 0.1 requires f to be **continuous** on R . The denominator $(x^2 + y^2)^2$ vanishes at the origin $(0, 0) \in R$, so f is *not even defined* there — let alone continuous — and the theorem does not apply.

[5]

Figure 3 shows the surface. The failure comes from the origin, where f is undefined and unbounded.

(This 3D surface is interactive; see the HTML or slide version of this page to view it.)

Figure 3: Surface $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ on $[0, 1]^2$, sampled away from the origin and clipped to $[-50, 50]$ for display. Approaching the origin, the function grows without bound along the x -axis (red ridge, $f > 0$ when $|x| > |y|$) and falls without bound along the y -axis (blue ridge, $f < 0$ when $|y| > |x|$). Because f is undefined at $(0, 0)$, f is not continuous on R and Theorem 0.1 does not apply.

Corollary 0.1 (Continuous functions on a rectangle (corollary of Theorem 0.1)). *If $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is **continuous** on the closed bounded rectangle $[a, b] \times [c, d]$, then:*

$$\begin{aligned}
\int_a^b \left(\int_c^d f(x, y) \, dy \right) dx &= \int_c^d \left(\int_a^b f(x, y) \, dx \right) dy \\
&= \iint_{[a, b] \times [c, d]} f(x, y) \, dA.
\end{aligned}$$

[3], Theorem 14.2, p. 982

i Proof

Proof. A closed bounded rectangle $[a, b] \times [c, d]$ is both vertically simple (with $g_1 \equiv c$, $g_2 \equiv d$) and horizontally simple (with $h_1 \equiv a$, $h_2 \equiv b$). Applying both parts of Theorem 0.1 to f on this rectangle gives the two iterated forms shown. $\square$

Example 0.5 (Evaluating a double integral on a rectangle). Structure adapted from [3, Sec. 14.2], Example 2, pp. 982–983; the integrand $x^2 + y^2$ is original, chosen so the integral equals $\mathbb{E}[g(X, Y)]$ for $g(x, y) = x^2 + y^2$.

Let X and Y be independent Uniform(0, 1) random variables, with joint density $f(x, y) = 1$ on the unit square $R = \{(x, y) : x \in [0, 1], y \in [0, 1]\}$ (Figure 4). Define the function $g(x, y) = x^2 + y^2$, and compute its expectation $\mathbb{E}[g(X, Y)]$.

Because the joint density equals 1 on R , this expectation is the double integral of g over R :

$$\mathbb{E}[g(X, Y)] = \iint_R (x^2 + y^2) dA.$$

```

ggplot2::ggplot() +
  ggplot2::annotate(
    "rect", xmin = 0, xmax = 1, ymin = 0, ymax = 1,
    fill = "steelblue", alpha = 0.4, color = "black", linewidth = 0.7
  ) +
  ggplot2::annotate(
    "text", x = 0.5, y = 0.5, label = "R", size = 8, fontface = "italic"
  ) +
  ggplot2::scale_x_continuous(breaks = c(0, 1), limits = c(-0.15, 1.25)) +
  ggplot2::scale_y_continuous(breaks = c(0, 1), limits = c(-0.15, 1.25)) +
  ggplot2::coord_equal() +
  ggplot2::labs(x = "First coordinate, x", y = "Second coordinate, y") +
  ggplot2::theme_minimal()

```

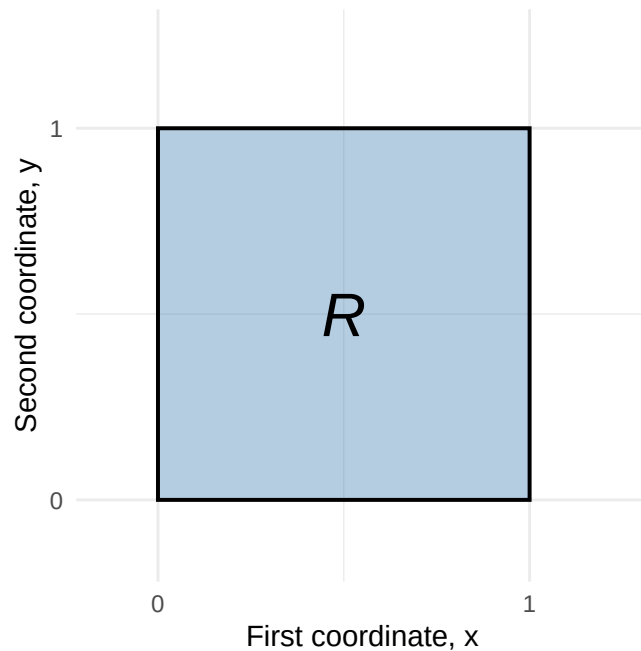


Figure 4: Integration region $R = [0, 1]^2$, the unit square.

The integrand is continuous on R , so Corollary 0.1 applies and either order of integration yields the same value.

Integrating y first, then x :

$$\begin{aligned}
\mathbb{E}[g(X, Y)] &= \int_0^1 \int_0^1 (x^2 + y^2) dy dx \\
&= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^1 dx \\
&= \int_0^1 \left(x^2 + \frac{1}{3} \right) dx \\
&= \left[\frac{x^3}{3} + \frac{x}{3} \right]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

Integrating x first, then y (verifying the order can be swapped):

$$\begin{aligned}
\int_0^1 \int_0^1 (x^2 + y^2) dx dy &= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_0^1 dy \\
&= \int_0^1 \left(\frac{1}{3} + y^2 \right) dy \\
&= \left[\frac{y}{3} + \frac{y^3}{3} \right]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

Both orders give $\frac{2}{3}$, as Corollary 0.1 guarantees.

As a cross-check, linearity of expectation gives the same value: since

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_0^1 x^2 dx \\
&= \frac{1}{3}
\end{aligned}$$

for $X \sim \text{Uniform}(0, 1)$ (and likewise for Y),

$$\begin{aligned}
\mathbb{E}[g(X, Y)] &= \mathbb{E}[X^2 + Y^2] \\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] \\
&= \frac{1}{3} + \frac{1}{3} \\
&= \frac{2}{3}.
\end{aligned}$$

The solid whose volume equals this integral is shown in Figure 5.

(This 3D surface is interactive; see the HTML or slide version of this page to view it.)

Figure 5: Surface $z = x^2 + y^2$ over the unit square $[0, 1]^2$. The double integral $\frac{2}{3}$ is the volume between this surface and the xy -plane, and equals $\mathbb{E}[X^2 + Y^2]$.

Exercise 0.3 (Double integral of a polynomial over a rectangle). Adapted from Miller [6], Question 1.1.47.

Evaluate the double integral:

$$\int_0^2 \int_0^3 5(x^2y + xy^2 + 2) dy dx$$

Solution

Solution 0.3. Integrate with respect to y first:

$$\begin{aligned} \int_0^3 (x^2y + xy^2 + 2) dy &= \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} + 2y \right]_{y=0}^3 \\ &= \frac{9}{2}x^2 + 9x + 6 \end{aligned}$$

Now integrate with respect to x :

$$\begin{aligned} 5 \int_0^2 \left(\frac{9}{2}x^2 + 9x + 6 \right) dx &= 5 \left[\frac{3}{2}x^3 + \frac{9}{2}x^2 + 6x \right]_0^2 \\ &= 5 \left[\frac{3}{2}(8) + \frac{9}{2}(4) + 6(2) \right] \\ &= 5[12 + 18 + 12] \\ &= 5(42) \\ &= 210 \end{aligned}$$

(Note: The source text carried an arithmetic slip in the inner coefficient resulting in 190; the exact value is 210.)

Exercise 0.4 (Choosing the order of integration). Adapted from Miller [6], Question 1.1.48.

Evaluate the double integral:

$$\int_0^6 \int_0^5 xe^{-xy} dy dx$$

Solution

Solution 0.4. Integrating with respect to x first requires integration by parts. Integrating with respect to y first is much simpler because the factor of x is already present:

$$\begin{aligned}\int_0^5 x e^{-xy} dy &= [-e^{-xy}]_{y=0}^5 \\ &= 1 - e^{-5x}\end{aligned}$$

Now integrate with respect to x :

$$\begin{aligned}\int_0^6 (1 - e^{-5x}) dx &= \left[x + \frac{1}{5} e^{-5x} \right]_0^6 \\ &= \left(6 + \frac{1}{5} e^{-30} \right) - \left(0 + \frac{1}{5} \right) \\ &= \frac{29}{5} + \frac{e^{-30}}{5}\end{aligned}$$

Exercise 0.5 (Product of single integrals for separable functions). Adapted from Miller [6], Question 1.1.49.

Let $m, n > 0$. Evaluate:

$$\int_0^1 \int_0^1 x^m y^n dy dx$$

Solution

Solution 0.5. Because the region of integration is a rectangle $[0, 1] \times [0, 1]$ and the integrand factors as $g(x)h(y) = x^m \cdot y^n$, the double integral splits into the product of two single-variable integrals:

$$\int_0^1 \int_0^1 x^m y^n dy dx = \left(\int_0^1 x^m dx \right) \left(\int_0^1 y^n dy \right)$$

Evaluating each factor:

$$\int_0^1 x^m dx = \frac{1}{m+1}, \quad \int_0^1 y^n dy = \frac{1}{n+1}$$

Therefore:

$$\int_0^1 \int_0^1 x^m y^n dy dx = \frac{1}{(m+1)(n+1)}$$

Remark: When joint random variables X and Y are independent, their joint density factors as $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, and joint probabilities over product sets factor in exactly this way.

Theorem 0.2 (Double integral of a product of powers). For $m, n > 0$,

$$\int_0^1 \int_0^1 x^m y^n dy dx = \frac{1}{(m+1)(n+1)}$$

Proof

Proof. This is the solution to Exercise 0.5, worked in Solution 0.5. □

Exercise 0.6 (Double integral with polynomial and radical terms). Adapted from Miller [6], Question 1.1.52.

Evaluate the double integral:

$$\int_0^1 \int_0^1 (x^2 + 2xy + y\sqrt{x}) dy dx$$

Solution

Solution 0.6. Integrate with respect to y first:

$$\begin{aligned} \int_0^1 (x^2 + 2xy + y\sqrt{x}) dy &= \left[x^2 y + xy^2 + \frac{y^2 \sqrt{x}}{2} \right]_{y=0}^1 \\ &= x^2 + x + \frac{1}{2} x^{1/2} \end{aligned}$$

Now integrate with respect to x :

$$\begin{aligned} \int_0^1 \left(x^2 + x + \frac{1}{2} x^{1/2} \right) dx &= \left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{2} \cdot \frac{x^{3/2}}{3/2} \right]_0^1 \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{3} x^{3/2} \right]_0^1 \\ &= \frac{1}{3} + \frac{1}{2} + \frac{1}{3} \\ &= \frac{2}{3} + \frac{1}{2} \\ &= \frac{7}{6} \end{aligned}$$

Exercise 0.7 (Double integral of an affine function). Adapted from Miller [6], Question 1.1.53.

Let a, b, c be constants. Evaluate:

$$\int_0^1 \int_0^1 (ax + by + c) dy dx$$

Solution

Solution 0.7. By linearity of the integral:

$$\begin{aligned} \int_0^1 \int_0^1 (ax + by + c) dy dx &= a \int_0^1 x dx \int_0^1 1 dy + b \int_0^1 1 dx \int_0^1 y dy + c \int_0^1 1 dx \int_0^1 1 dy \\ &= a \left(\frac{1}{2} \right) (1) + b(1) \left(\frac{1}{2} \right) + c(1)(1) \\ &= \frac{a}{2} + \frac{b}{2} + c \end{aligned}$$

Exercise 0.8 (A double integral with a parameter). Adapted from Miller [6], Question 1.1.54.

Evaluate the double integral

$$\int_0^1 \int_0^1 n(1 - xy)^{n-1} dx dy$$

for every positive integer n , and write the result as a sum of reciprocals of integers.

Solution

Solution 0.8. Integration, then a geometric series: Integrate with respect to x first:

$$\begin{aligned} \int_0^1 n(1 - xy)^{n-1} dx &= \left[-\frac{(1 - xy)^n}{y} \right]_{x=0}^1 \\ &= \frac{1 - (1 - y)^n}{y} \end{aligned}$$

Use the finite geometric series identity $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$ with $r = 1 - y$:

$$\frac{1 - (1 - y)^n}{y} = \sum_{k=0}^{n-1} (1 - y)^k$$

Now integrate each term with respect to y from 0 to 1:

$$\begin{aligned}\int_0^1 (1-y)^k dy &= \left[-\frac{(1-y)^{k+1}}{k+1} \right]_0^1 \\ &= \frac{1}{k+1}\end{aligned}$$

Summing over $k = 0, 1, \dots, n-1$:

$$\begin{aligned}\sum_{k=0}^{n-1} \frac{1}{k+1} &= \sum_{j=1}^n \frac{1}{j} \\ &= 1 + \frac{1}{2} + \dots + \frac{1}{n}\end{aligned}$$

The result is stated as Theorem 0.3.

Remark (binomial expansion): The same integral can be expanded in a different way. Expand $(1-xy)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (xy)^k$. Integrating over $[0, 1]^2$:

$$\int_0^1 \int_0^1 (xy)^k dx dy = \left(\frac{1}{k+1} \right)^2$$

Multiplying by n and using the identity $\frac{n}{k+1} \binom{n-1}{k} = \binom{n}{k+1}$:

$$\int_0^1 \int_0^1 n(1-xy)^{n-1} dx dy = \sum_{k=0}^{n-1} \binom{n}{k+1} \frac{(-1)^k}{k+1}$$

This reaches the alternating binomial sum for the same integral as above, so it inherits the value $\sum_{j=1}^n \frac{1}{j}$ from the integration-then-geometric-series computation.

Remark: This integral identity connects multivariable integration with harmonic numbers $H_n = \sum_{k=1}^n \frac{1}{k} \approx \log n + \gamma$, where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

Theorem 0.3 (Harmonic numbers as a double integral). *For every positive integer n ,*

$$\int_0^1 \int_0^1 n(1-xy)^{n-1} dx dy = \sum_{k=1}^n \frac{1}{k}$$

Proof

Proof. This is the solution to Exercise 0.8, worked in Solution 0.8. □

Theorem 0.4 (Tonelli's theorem (measure-theoretic form)). Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be *measure spaces* with *σ -finite* measures, and let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be *measurable* with respect to the *product σ -algebra* $\mathcal{F}_1 \otimes \mathcal{F}_2$. If $f \geq 0$ *almost everywhere* with respect to the *product measure* $\mu_1 \otimes \mu_2$, then both iterated *integrals* exist (possibly equal to $+\infty$), agree with the double integral, and equal each other:

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \otimes \mu_2) &= \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) d\mu_2(\omega_2) \right) d\mu_1(\omega_1) \\ &= \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) d\mu_1(\omega_1) \right) d\mu_2(\omega_2). \end{aligned}$$

[7], Theorem 18.3, [8], Theorem 9.1, p. 65, [5]

Theorem 0.5 (Fubini's theorem (measure-theoretic form)). Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be *measure spaces* with *σ -finite* measures, and let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be *measurable* with respect to the *product σ -algebra* $\mathcal{F}_1 \otimes \mathcal{F}_2$. If $\int_{\Omega_1 \times \Omega_2} |f| d(\mu_1 \otimes \mu_2) < \infty$, that is, f is *absolutely integrable*, then both iterated *integrals* exist, agree with the double integral, and equal each other:

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \otimes \mu_2) &= \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) d\mu_2(\omega_2) \right) d\mu_1(\omega_1) \\ &= \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) d\mu_1(\omega_1) \right) d\mu_2(\omega_2). \end{aligned}$$

[7], Theorem 18.3, [8], Theorem 9.1, p. 65, [5], [9]

i Remark (Fubini–Tonelli for probability measures)

Remark 0.1 (Fubini–Tonelli for probability measures). Applied courses rarely need the measure-theoretic generalization itself, but it is what justifies the *joint-distribution form* corollary in *Probability for Data Science*. A *probability measure* P on Ω has $P(\Omega) = 1 < \infty$, so it is *finite*, and hence *σ -finite* (*finite and σ -finite measures*); for probability measures, the *σ -finiteness* condition is automatic.

The integrability conditions (nonnegativity or *absolute integrability*) still need to be verified in each application. For example, *Lebesgue measure* (ordinary length) on $[0, 1]$ is a probability measure, so the *σ -finiteness* condition holds for both factors $[0, 1]$, yet the two iterated integrals in Example 0.4 are $\pi/4$ and $-\pi/4$. So *σ -finiteness* alone does not make the iterated integrals agree.

Example 0.6 (Positive application of Theorem 0.4). Let X and Y be [independent Exponential\(1\) random variables](#), with [joint density](#) $f(x, y) = e^{-(x+y)}$ for $x, y \geq 0$.

The probability $P(X \leq 1, Y \leq 1)$ is the integral of f over $[0, 1]^2$ with respect to Lebesgue measure (ordinary length) in each coordinate. Lebesgue measure on $[0, \infty)$ is σ -finite, because $[0, \infty)$ is the union of the intervals $[0, n]$, $n \in \mathbb{N}$, each of finite length n ; so the σ -finiteness condition of Theorem 0.4 holds. Since $f(x, y) = e^{-(x+y)} \geq 0$, the nonnegativity condition is also satisfied.

By Theorem 0.4, both iterated integrals exist and agree. Integrating y first, then x :

$$\begin{aligned}
 P(X \leq 1, Y \leq 1) &= \int_0^1 \int_0^1 e^{-(x+y)} dy dx \\
 &= \int_0^1 \int_0^1 e^{-x} e^{-y} dy dx && \text{(exponential of a sum)} \\
 &= \int_0^1 e^{-x} \left(\int_0^1 e^{-y} dy \right) dx && (e^{-x} \text{ is constant in } y) \\
 &= \int_0^1 e^{-x} [-e^{-y}]_{y=0}^{y=1} dx && \text{(antiderivative of } e^{-y}) \\
 &= \int_0^1 e^{-x} (1 - e^{-1}) dx && \text{(evaluate at the limits)} \\
 &= (1 - e^{-1}) \int_0^1 e^{-x} dx && (1 - e^{-1} \text{ is constant in } x) \\
 &= (1 - e^{-1}) [-e^{-x}]_{x=0}^{x=1} && \text{(antiderivative of } e^{-x}) \\
 &= (1 - e^{-1})^2 && \text{(evaluate at the limits)}
 \end{aligned}$$

Integrating x first, then y :

$$\begin{aligned}
P(X \leq 1, Y \leq 1) &= \int_0^1 \int_0^1 e^{-(x+y)} dx dy \\
&= \int_0^1 \int_0^1 e^{-y} e^{-x} dx dy && \text{(exponential of a sum)} \\
&= \int_0^1 e^{-y} \left(\int_0^1 e^{-x} dx \right) dy && (e^{-y} \text{ is constant in } x) \\
&= \int_0^1 e^{-y} [-e^{-x}]_{x=0}^{x=1} dy && \text{(antiderivative of } e^{-x}) \\
&= \int_0^1 e^{-y} (1 - e^{-1}) dy && \text{(evaluate at the limits)} \\
&= (1 - e^{-1}) \int_0^1 e^{-y} dy && (1 - e^{-1} \text{ is constant in } y) \\
&= (1 - e^{-1}) [-e^{-y}]_{y=0}^{y=1} && \text{(antiderivative of } e^{-y}) \\
&= (1 - e^{-1})^2 && \text{(evaluate at the limits)}
\end{aligned}$$

Both iterated integrals equal $(1 - e^{-1})^2 \approx 0.400$, as Theorem 0.4 guarantees.

Example 0.7 (When neither Tonelli's nor Fubini's condition is satisfied). The same function $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ from Example 0.4 illustrates a case where neither the hypothesis of Theorem 0.4 nor that of Theorem 0.5 is satisfied.

Why the hypotheses fail: $\iint_R |f| dA = \infty$, which violates the absolute-integrability hypothesis of Theorem 0.5. Switching to polar coordinates (r, θ) near the origin, the integrand satisfies

$$\begin{aligned}
|f(x, y)| &= |x^2 - y^2|/(x^2 + y^2)^2 \\
&= |\cos 2\theta|/r^2,
\end{aligned}$$

so

$$\begin{aligned}
\iint_R |f| dA &\geq \int_0^{\pi/2} \int_0^\varepsilon \frac{|\cos 2\theta|}{r^2} r dr d\theta \\
&= \left(\int_0^{\pi/2} |\cos 2\theta| d\theta \right) \int_0^\varepsilon \frac{dr}{r} \\
&= +\infty,
\end{aligned}$$

since $\int_0^\varepsilon dr/r$ diverges. Therefore $\iint_R |f| dA = \infty$, and the hypothesis of Theorem 0.5 is not satisfied. (The hypothesis of Theorem 0.4 also fails: f takes both positive and negative values, so it is not nonnegative a.e.) The unequal iterated integrals from Example 0.4 are

thus consistent with both theorems: neither one applies.
[5]

Further reading

- Wikipedia contributors [5] — Fubini’s theorem, in the Riemann and Lebesgue settings
- [1] Wikipedia contributors, “Tonelli’s theorem (functional analysis) — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Tonelli%27s_theorem_%28functional_analysis%29
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 - [4] Wikipedia contributors, “Iterated integral — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Iterated_integral
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 - [9] G. Fubini, “Sugli integrali multipli,” *Rendiconti della Reale Accademia dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 16, pp. 608–614, 1907.