

Derivatives and Taylor Series

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Derivatives

Limits and derivatives

Definition 0.1 (Limit of a function at a point). Let f be a [function](#) defined at every point of an [open interval](#) around c , except possibly at c itself, and let L be a [real number](#). The **limit** of $f(x)$ as x approaches c is L [1], written $\lim_{x \rightarrow c} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for every } x \text{ with } 0 < |x - c| < \delta.$$

Here $|\cdot|$ is the [absolute value](#). When such a real number L exists, the limit **exists**; otherwise, the limit does not exist. The value $f(c)$, if it is defined, plays no role.

Example 0.1 (The limit of $3x + 1$ at 2). $\lim_{x \rightarrow 2} (3x + 1) = 7$. Given $\varepsilon > 0$, take $\delta = \varepsilon/3$. For every x with $0 < |x - 2| < \delta$, using the [distributive law](#) in the second line,

$$\begin{aligned} |(3x + 1) - 7| &= |3x - 6| && \text{(subtract)} \\ &= |3(x - 2)| && \text{(distributive law)} \\ &= 3|x - 2| && (|3y| = 3|y|, \text{ since } 3 > 0) \\ &< 3 \cdot \frac{\varepsilon}{3} && (|x - 2| < \delta = \varepsilon/3) \\ &= \varepsilon && \text{(multiply)} \end{aligned}$$

For example, with $\varepsilon = 0.3$ and $\delta = 0.1$, the point $x = 2.05$ satisfies $0 < |2.05 - 2| < 0.1$, and

$$\begin{aligned} |(3 \cdot 2.05 + 1) - 7| &= |7.15 - 7| \\ &= 0.15 \\ &< 0.3. \end{aligned}$$

Definition 0.2 (Right-hand limit). Let f be a function and L a real number. The **right-hand limit** of f at c is L [2], written $\lim_{x \rightarrow c^+} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for every x with $c < x < c + \delta$.

Definition 0.3 (Left-hand limit). Let f be a function and L a real number. The **left-hand limit** of f at c is L [2], written $\lim_{x \rightarrow c^-} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for every x with $c - \delta < x < c$.

Definition 0.4 (One-sided limits). Let f be a function. The right-hand limit (Definition 0.2) and the left-hand limit (Definition 0.3) of f at c are the **one-sided limits** of f at c .

Example 0.2 (One-sided limits of a step function). Let $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$.

- $\lim_{x \rightarrow 0^+} H(x) = 1$, because $H(x) = 1$ for every $x > 0$.
- $\lim_{x \rightarrow 0^-} H(x) = 0$, because $H(x) = 0$ for every $x < 0$.

The one-sided limits differ, since $1 \neq 0$, so $\lim_{x \rightarrow 0} H(x)$ does not exist.

i Remark (A limit exists exactly when both one-sided limits exist and are equal)

Remark 0.1 (A limit exists exactly when both one-sided limits exist and are equal). The limit $\lim_{x \rightarrow c} f(x)$ (Definition 0.1) exists exactly when both one-sided limits (Definition 0.4) exist and are equal, and then all three are equal.

Videos

Jon Krohn's "Calculus for Machine Learning" YouTube playlist introduces limits and derivatives:

- [Calculating Limits](#)
- [Exercises on Limits](#)
- [Intro to Differential Calculus](#)
- [How Derivatives Arise from Limits](#)
- [Derivative Notation](#)

Definition 0.5 (Difference quotient). Let f be a real-valued function of one variable, defined at c and at $c + h$, with $h \neq 0$. The **difference quotient** of f at c with increment

h is [3]

$$\frac{f(c+h) - f(c)}{h},$$

the **slope** of the line through the points $(c, f(c))$ and $(c+h, f(c+h))$.

Example 0.3 (The difference quotient of x^2 at 1). For $f(x) = x^2$, $c = 1$, and $h \neq 0$,

$$\begin{aligned}\frac{f(1+h) - f(1)}{h} &= \frac{(1+h)^2 - 1}{h} && \text{(substitute into } f) \\ &= \frac{1 + 2h + h^2 - 1}{h} && \text{(expand } (1+h)^2) \\ &= \frac{2h + h^2}{h} && (1 - 1 = 0) \\ &= 2 + h, && \text{(divide by } h \neq 0)\end{aligned}$$

which tends to 2 as $h \rightarrow 0$ (Definition 0.1). With $h = 0.1$ the difference quotient is $2 + 0.1 = 2.1$: the line through $(1, 1)$ and $(1.1, 1.21)$ has slope

$$\begin{aligned}\frac{1.21-1}{0.1} &= \frac{0.21}{0.1} \\ &= 2.1.\end{aligned}$$

Definition 0.6 (Differentiable function). A function f is **differentiable at** $x = c$ [4] if the limit (Definition 0.1) of the difference quotient (Definition 0.5) as $h \rightarrow 0$,

$$f'(c) \stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h},$$

exists and is finite. The value of this limit is denoted $f'(c)$.

[5, Sec. 2.1], p. 100

Example 0.4 (A differentiable function, and a limit that is not finite).

- $f(x) = x^2$ is differentiable at $c = 3$:

$$\begin{aligned}
\frac{f(3+h) - f(3)}{h} &= \frac{(3+h)^2 - 3^2}{h} && \text{(substitute } f(x) = x^2\text{)} \\
&= \frac{9 + 6h + h^2 - 9}{h} && \text{(expand } (3+h)^2\text{)} \\
&= \frac{6h + h^2}{h} && \text{(cancel } 9 - 9\text{)} \\
&= 6 + h && \text{(divide by } h \neq 0\text{)}
\end{aligned}$$

which tends to the finite limit $f'(3) = 6$ as $h \rightarrow 0$.

- $g(x) = \sqrt[3]{x}$ is not differentiable at $c = 0$: the difference quotient

$$\begin{aligned}
\frac{g(h) - g(0)}{h} &= \frac{\sqrt[3]{h}}{h} \\
&= \frac{1}{(\sqrt[3]{h})^2}
\end{aligned}$$

grows without bound as $h \rightarrow 0$, so the limit is not finite.

Definition 0.7 (Derivative). If f is differentiable at c (Definition 0.6), the number $f'(c)$ is the **derivative** of f at c [6]. The **derivative** of f is the function f' whose value at each x where f is differentiable is $f'(x)$.

Definition 0.8 (Lagrange's notation). Writing the derivative (Definition 0.7) with a prime, as f' or $f'(x)$, is **Lagrange's notation** [7].

Definition 0.9 (Leibniz notation). The derivative of f (Definition 0.7) is also written $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$, which is **Leibniz notation** [7].

Example 0.5 (The derivative of x^2). For $f(x) = x^2$, the computation in Example 0.4, with 3 replaced by any real number c , gives $\frac{f(c+h) - f(c)}{h} = 2c + h$, which tends to $2c$ as $h \rightarrow 0$. So the derivative of f is $f'(x) = 2x$. For example, $f'(3) = 6$ and $f'(-1) = -2$.

Definition 0.10 (Right-hand derivative). Let f be a function defined at c . The **right-hand derivative** of f at c is the right-hand limit (Definition 0.2) $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$ of the difference quotient (Definition 0.5), when that limit exists and is finite.

Definition 0.11 (Left-hand derivative). Let f be a function defined at c . The **left-hand derivative** of f at c is the left-hand limit (Definition 0.3) $\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h}$, when that

limit exists and is finite.

Definition 0.12 (One-sided derivatives). Let f be a function defined at c . The right-hand derivative (Definition 0.10) and the left-hand derivative (Definition 0.11) of f at c are the **one-sided derivatives** of f at c .

Example 0.6 (One-sided derivatives of $|x|$ at 0). Let $f(x) = |x|$, the **absolute value**, and $c = 0$.

- For $h > 0$, $|h| = h$, so

$$\begin{aligned}\frac{f(0+h) - f(0)}{h} &= \frac{h - 0}{h} \\ &= 1.\end{aligned}$$

The right-hand derivative of f at 0 is 1.

- For $h < 0$, $|h| = -h$, so

$$\begin{aligned}\frac{f(0+h) - f(0)}{h} &= \frac{-h - 0}{h} \\ &= -1.\end{aligned}$$

The left-hand derivative of f at 0 is -1 .

For example, $h = 0.5$ gives $\frac{0.5}{0.5} = 1$, and $h = -0.5$ gives $\frac{0.5}{-0.5} = -1$. The one-sided derivatives differ, since $1 \neq -1$.

Definition 0.13 (Second derivative). Let f be a real-valued function of one variable whose derivative f' (Definition 0.7) exists at every point of an **open interval** containing c . If f' is differentiable (Definition 0.6) at c , its derivative there, written $f''(c)$ or $\frac{d^2f}{dx^2}$, is the **second derivative** of f at c [8].

Example 0.7 (The second derivative of x^3). For $f(x) = x^3$, the power rule (Theorem 0.3) gives $f'(x) = 3x^2$ at every x , and differentiating again, with the constant multiple rule (Theorem 0.2), gives

$$\begin{aligned}f''(x) &= 3 \cdot 2x \\ &= 6x.\end{aligned}$$

So $f''(2) = 12$, $f''(0) = 0$, and $f''(-1) = -6$.

Derivative rules

Theorem 0.1 (Constant rule). *If c is constant with respect to x , then*

$$\frac{\partial}{\partial x} c = 0$$

Video

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist has a video on this rule:

- [The Derivative of a Constant](#)

Theorem 0.2 (Constant multiple rule). *If a is constant with respect to x and y is a differentiable function of x , then:*

$$\frac{\partial}{\partial x} ay = a \frac{\partial y}{\partial x}$$

Video

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist has a video on this rule:

- [The Constant Multiple Rule for Derivatives](#)

Theorem 0.3 (Power rule). *For every real number q and every $x > 0$ [9], the derivative of the [power](#) x^q is:*

$$\frac{\partial}{\partial x} x^q = qx^{q-1}$$

When q is a positive [integer](#), the same formula holds for every real x .

Videos

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist has videos on this rule, on the sum rule (the derivative of a sum is the sum of the derivatives), and on exercises that combine the rules so far:

- [The Power Rule for Derivatives](#)
- [The Sum Rule for Derivatives](#)
- [Exercises on Derivative Rules](#)

Exercise 0.1 (Derivative of polynomial and power functions). Adapted from Miller [10], Question 1.1.1.
Find the derivative of

$$f(x) = 4x^5 + 3x^2 + x^{1/3}$$

Solution

Solution 0.1. Differentiate term by term using the constant multiple rule (Theorem 0.2) and the power rule (Theorem 0.3):

$$\begin{aligned}\frac{d}{dx}(4x^5) &= 4 \cdot (5x^4) \\ &= 20x^4\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(3x^2) &= 3 \cdot (2x) \\ &= 6x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(x^{1/3}) &= \frac{1}{3}x^{1/3-1} \\ &= \frac{1}{3}x^{-2/3}\end{aligned}$$

Combining the three terms yields:

$$f'(x) = 20x^4 + 6x + \frac{1}{3}x^{-2/3}$$

Exercise 0.2 (Finding points with a prescribed slope). Adapted from Miller [10], Question 1.1.14.

Let $f(x) = \frac{1}{3}x^3 + x^2 - x - 1$. Find all points on the graph of f where the slope is:

1. -1
2. 2
3. 0

Solution

Solution 0.2. The slope is given by the derivative:

$$f'(x) = x^2 + 2x - 1$$

1. Slope -1 : Solve

$$\begin{aligned}x^2 + 2x - 1 = -1 &\iff x^2 + 2x = 0 \\&\iff x(x + 2) = 0.\end{aligned}$$

The roots are $x = 0$ and $x = -2$. Evaluating f : $f(0) = -1$ and

$$\begin{aligned}f(-2) &= -\frac{8}{3} + 4 + 2 - 1 \\&= \frac{7}{3}.\end{aligned}$$

The points are $(0, -1)$ and $(-2, 7/3)$.

2. Slope 2 : Solve

$$\begin{aligned}x^2 + 2x - 1 = 2 &\iff x^2 + 2x - 3 = 0 \\&\iff (x + 3)(x - 1) = 0.\end{aligned}$$

The roots are $x = -3$ and $x = 1$. Evaluating f :

$$\begin{aligned}f(-3) &= -9 + 9 + 3 - 1 \\&= 2\end{aligned}$$

and

$$\begin{aligned}f(1) &= \frac{1}{3} + 1 - 1 - 1 \\&= -\frac{2}{3}.\end{aligned}$$

The points are $(-3, 2)$ and $(1, -2/3)$.

3. Slope 0 : Solve $x^2 + 2x - 1 = 0$. By the quadratic formula:

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2} \\&= \frac{-2 \pm \sqrt{8}}{2} \\&= -1 \pm \sqrt{2}\end{aligned}$$

Theorem 0.4 (Derivative of natural logarithm). *For every $x > 0$, the derivative of the natural logarithm is:*

$$\begin{aligned}\log' \{x\} &= \frac{1}{x} \\ &= x^{-1}\end{aligned}$$

Exercise 0.3 (Derivative after logarithmic simplification). Adapted from Miller [10], Question 1.1.4.

Find the derivative of

$$f(x) = \log(4x) - \log(2x)$$

for $x > 0$.

Solution

Solution 0.3. Simplify before differentiating using the quotient property of logarithms:

$$\begin{aligned}f(x) &= \log(4x) - \log(2x) \\ &= \log\left(\frac{4x}{2x}\right) \\ &= \log 2\end{aligned}$$

Because $\log 2$ is constant with respect to x , the constant rule (Theorem 0.1) gives:

$$f'(x) = 0$$

Remark: While one could apply the chain rule separately to each term, giving

$$\begin{aligned}\frac{4}{4x} - \frac{2}{2x} &= \frac{1}{x} - \frac{1}{x} \\ &= 0,\end{aligned}$$

simplifying algebraically first is faster and prevents arithmetic errors.

Theorem 0.5 (Derivative of exponential). *For every real x , the derivative of the exponential function is:*

$$\exp' \{x\} = \exp \{x\}$$

Theorem 0.6 (Derivative of sine). *For every real x , the derivative of the sine function is:*

$$\frac{\partial}{\partial x} \sin x = \cos x$$

Theorem 0.7 (Derivative of cosine). *For every real x , the derivative of the cosine function is:*

$$\frac{\partial}{\partial x} \cos x = -\sin x$$

Theorem 0.8 (Product rule). *If a and b are differentiable functions of x [11], then*

$$(ab)' = ab' + ba'$$

Video

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist has a video on this rule:

- [The Product Rule for Derivatives](#)

Exercise 0.4 (Derivative of a product). Adapted from Miller [10], Question 1.1.2.
Find the derivative of

$$f(x) = (x^4 + 3x^2 + 8) \cos x$$

Solution

Solution 0.4. Apply the product rule (Theorem 0.8) with:

$$u(x) = x^4 + 3x^2 + 8, \quad v(x) = \cos x$$

Their derivatives are:

$$u'(x) = 4x^3 + 6x, \quad v'(x) = -\sin x$$

By the product rule:

$$\begin{aligned} f'(x) &= u'(x)v(x) + u(x)v'(x) \\ &= (4x^3 + 6x) \cos x - (x^4 + 3x^2 + 8) \sin x \end{aligned}$$

Exercise 0.5 (Second derivative of a product). Adapted from Miller [10], Question 1.1.15. Find the second derivative of

$$f(x) = (x^4 + 3x^2 + 8) \cos x$$

Solution

Solution 0.5. From Exercise 0.4, the first derivative is:

$$f'(x) = (4x^3 + 6x) \cos x - (x^4 + 3x^2 + 8) \sin x$$

Differentiate each term via the product rule:

$$\begin{aligned} \frac{d}{dx}[(4x^3 + 6x) \cos x] &= (12x^2 + 6) \cos x - (4x^3 + 6x) \sin x \\ \frac{d}{dx}[-(x^4 + 3x^2 + 8) \sin x] &= -(4x^3 + 6x) \sin x - (x^4 + 3x^2 + 8) \cos x \end{aligned}$$

Combining and grouping like trigonometric terms:

$$\begin{aligned} f''(x) &= [(12x^2 + 6) - (x^4 + 3x^2 + 8)] \cos x - 2(4x^3 + 6x) \sin x \\ &= (-x^4 + 9x^2 - 2) \cos x - (8x^3 + 12x) \sin x \end{aligned}$$

Theorem 0.9 (Quotient rule). *If a and b are differentiable functions of x [12] and $b \neq 0$, then*

$$(a/b)' = a'/b - (a/b^2)b'$$

 Video

Jon Krohn's "Calculus for Machine Learning" YouTube playlist has a video on this rule:

- [The Quotient Rule for Derivatives](#)

Exercise 0.6 (Derivative of a rational function). Adapted from Miller [10], Question 1.1.12.

Find the derivative of

$$f(x) = \frac{x^2 - 1}{x - 1}$$

for $x \neq 1$.

Solution

Solution 0.6. Factor the numerator before differentiating:

$$\begin{aligned} f(x) &= \frac{(x-1)(x+1)}{x-1} \\ &= x+1 \quad (x \neq 1) \end{aligned}$$

Differentiating directly gives $f'(x) = 1$.

Alternative (quotient rule): Using Theorem 0.9 with $u(x) = x^2 - 1$ and $v(x) = x - 1$:

$$\begin{aligned} f'(x) &= \frac{2x(x-1) - (x^2-1)(1)}{(x-1)^2} \\ &= \frac{x^2 - 2x + 1}{(x-1)^2} \\ &= \frac{(x-1)^2}{(x-1)^2} \\ &= 1 \end{aligned}$$

Both methods yield 1, but factoring first eliminates tedious algebraic simplification.

Theorem 0.10 (Chain rule). *If a is a differentiable function of b [13], and b is a differentiable function of c , then a is a differentiable function of c , and*

$$\begin{aligned} \frac{da}{dc} &= \frac{da}{db} \frac{db}{dc} \\ &= \frac{db}{dc} \frac{da}{db} \end{aligned}$$

or in Lagrange's notation (Definition 0.7), if g is differentiable at x and f is differentiable at $g(x)$, then for the composition $f \circ g$, with inner function g and outer function f :

$$(f(g(x)))' = g'(x)f'(g(x))$$

Exercise 0.7 (Derivative of the standard Gaussian kernel). Adapted from Miller [10], Question 1.1.5.

Find the derivative of

$$f(x) = \exp\left\{-\frac{x^2}{2}\right\}$$

Solution

Solution 0.7. Apply the chain rule (Theorem 0.10) with outer function $\exp\{u\}$ and inner function $u(x) = -x^2/2$. Since $\frac{d}{du}\exp\{u\} = \exp\{u\}$ (Theorem 0.5) and $u'(x) = -x$:

$$\begin{aligned} f'(x) &= u'(x)\exp\{u(x)\} \\ &= -x\exp\left\{-\frac{x^2}{2}\right\} \end{aligned}$$

Theorem 0.11 (Derivative of the standard Gaussian kernel). *For $f(x) = \exp\left\{-\frac{x^2}{2}\right\}$ and every $x \in \mathbb{R}$,*

$$f'(x) = -x\exp\left\{-\frac{x^2}{2}\right\}$$

i Proof

Proof. This is the solution to Exercise 0.7, worked in Solution 0.7. □

Exercise 0.8 (Second derivative of the Gaussian kernel). Adapted from Miller [10], Question 1.1.6.

Find the second derivative of

$$f(x) = \exp\left\{-\frac{x^2}{2}\right\}$$

Solution

Solution 0.8. From Theorem 0.11, the first derivative is $f'(x) = -x\exp\{-x^2/2\}$. Apply the product rule (Theorem 0.8) to $u(x) = -x$ and $v(x) = \exp\{-x^2/2\}$:

$$\begin{aligned} u'(x) &= -1 \\ v'(x) &= -x\exp\left\{-\frac{x^2}{2}\right\} \end{aligned}$$

Therefore:

$$\begin{aligned}
f''(x) &= u'(x)v(x) + u(x)v'(x) \\
&= (-1)\exp\left\{-\frac{x^2}{2}\right\} + (-x)\left[-x\exp\left\{-\frac{x^2}{2}\right\}\right] \\
&= (x^2 - 1)\exp\left\{-\frac{x^2}{2}\right\}
\end{aligned}$$

Theorem 0.12 (Second derivative of the standard Gaussian kernel). For $f(x) = \exp\left\{-\frac{x^2}{2}\right\}$ and every $x \in \mathbb{R}$,

$$f''(x) = (x^2 - 1)\exp\left\{-\frac{x^2}{2}\right\}$$

Proof

Proof. This is the solution to Exercise 0.8, worked in Solution 0.8. □

Exercise 0.9 (Combining product and chain rules). Adapted from Miller [10], Question 1.1.7.

Find the derivative of

$$f(x) = \exp\{x^8\} \cos(3x^4)$$

Solution

Solution 0.9. Apply the product rule (Theorem 0.8) to $u(x) = \exp\{x^8\}$ and $v(x) = \cos(3x^4)$:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Compute $u'(x)$ and $v'(x)$ via the chain rule (Theorem 0.10):

$$\begin{aligned}
u'(x) &= 8x^7 \exp\{x^8\} \\
v'(x) &= -12x^3 \sin(3x^4)
\end{aligned}$$

Substitute both derivatives into the product rule:

$$\begin{aligned}
f'(x) &= 8x^7 \exp\{x^8\} \cos(3x^4) - 12x^3 \exp\{x^8\} \sin(3x^4) \\
&= 4x^3 \exp\{x^8\} [2x^4 \cos(3x^4) - 3 \sin(3x^4)]
\end{aligned}$$

Exercise 0.10 (Generalized power rule). Adapted from Miller [10], Question 1.1.13.
Find the derivative of

$$\begin{aligned} f(x) &= \sqrt[3]{(5x-2)^2} \\ &= (5x-2)^{2/3} \end{aligned}$$

for $x \neq 2/5$.

Solution

Solution 0.10. Apply the generalized power rule $\frac{d}{dx}[g(x)]^r = r[g(x)]^{r-1}g'(x)$ with $g(x) = 5x - 2$ and $r = 2/3$:

$$\begin{aligned} f'(x) &= \frac{2}{3}(5x-2)^{2/3-1} \cdot 5 \\ &= \frac{10}{3}(5x-2)^{-1/3} \\ &= \frac{10}{3(5x-2)^{1/3}} \end{aligned}$$

Remark: Omitting the inner derivative $g'(x) = 5$ is a frequent mistake when applying the generalized power rule.

Corollary 0.1 (Chain rule for logarithms). *If f is differentiable at x and $f(x) > 0$, then*

$$\frac{d}{dx}\log\{f(x)\} = \frac{f'(x)}{f(x)}$$

i Proof

Proof. Apply Theorem 0.10 and Theorem 0.4:

$$\begin{aligned} \frac{d}{dx}\log\{f(x)\} &= f'(x) \cdot \log'\{f(x)\} \quad (\text{chain rule, with } g = f \text{ and outer function } \log) \\ &= f'(x) \cdot \frac{1}{f(x)} \quad (\text{derivative of } \log, \text{ valid because } f(x) > 0) \\ &= \frac{f'(x)}{f(x)} \quad (\text{multiply}) \end{aligned}$$

□

💡 Videos

Jon Krohn’s “Calculus for Machine Learning” YouTube playlist has videos on this rule and on exercises that combine it with the other rules:

- [The Chain Rule for Derivatives](#)
- [The Power Rule on a Function Chain](#)
- [Advanced Exercises on Derivative Rules](#)

Exercise 0.11 (Chain rule with logarithm). Adapted from Miller [10], Question 1.1.3. Find the derivative of

$$f(x) = \log(1 - x^2)$$

for $x \in (-1, 1)$.

Solution

Solution 0.11. Apply the chain rule for logarithms (Corollary 0.1): if $g(x) = 1 - x^2$, then $g'(x) = -2x$, and

$$\begin{aligned} f'(x) &= \frac{g'(x)}{g(x)} \\ &= -\frac{2x}{1 - x^2} \end{aligned}$$

Remark: A common pitfall in chain rule problems is evaluating the outer derivative at x rather than at the inner value $g(x)$. The denominator is $g(x) = 1 - x^2$, not x .

Exercise 0.12 (Second derivative with an added constant). Adapted from Miller [10], Question 1.1.9.

Find the second derivative of

$$f(x) = \log x + \sqrt{162}$$

for $x > 0$.

Solution

Solution 0.12. Because $\sqrt{162}$ is constant with respect to x , its derivative is zero (Theorem 0.1):

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ &= x^{-1} \end{aligned}$$

Differentiating again via the power rule (Theorem 0.3):

$$\begin{aligned} f''(x) &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

Studying a new function

Definition 0.14 (Logarithmic derivative). The **logarithmic derivative** $\frac{f'(x)}{f(x)}$ of a function f of a real variable equals $\frac{d}{dx} \log\{f(x)\}$ by Corollary 0.1 and measures the rate of change of f relative to its size; it needs $f(x) > 0$, and for $f(x) < 0$ the same expression is the derivative of $\log\{|f(x)|\}$.

Definition 0.15 (Profile of a function). The **profile** of a function f of a real variable is the list of related functions and values to work out when meeting f for the first time:

- the logarithm $\log\{f(x)\}$, where $f(x) > 0$;
- the derivatives $f'(x)$, $f''(x)$, and so on, as far as they are informative;
- the **logarithmic derivative** $\frac{f'(x)}{f(x)}$;
- the inverse function f^{-1} (**definition**), which exists on the **image** of f when f is one-to-one, and otherwise only after restricting the domain of f ;
- the complement $1 - f(x)$, which is informative when f takes values in $[0, 1]$, as a probability does;
- the reciprocal $\frac{1}{f(x)}$, where $f(x) \neq 0$ (not to be confused with f^{-1});
- an antiderivative of f , a function F with $F' = f$ (**integration**);
- the values $f(0)$ and $f(1)$, and the limits of $f(x)$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, at each of these that lies in the domain of f or exists;
- the graph of f .

Example 0.8 (The profile of the logistic function). The **logistic function** (also called expit or the sigmoid) is

$$f(x) = \frac{1}{1 + e^{-x}}.$$

The profile in Definition 0.15 of f is as follows. Write $E = e^{-x}$, so that $f = \frac{1}{1+E}$ and $1 - f = \frac{E}{1+E}$.

- **Logarithm:** $\log\{f(x)\} = -\log\{1 + e^{-x}\}$, defined for every real x because $f(x) > 0$.
- **Derivative:** by the chain rule,

$$\begin{aligned} f'(x) &= \frac{e^{-x}}{(1 + e^{-x})^2} && \text{(chain rule applied to } (1 + e^{-x})^{-1} \text{)} \\ &= \frac{1}{1 + E} \cdot \frac{E}{1 + E} && \text{(split the square)} \\ &= f(x)(1 - f(x)). && \text{(identify } f \text{ and } 1 - f \text{)} \end{aligned}$$

Differentiating again with the product rule gives $f''(x) = f'(1 - f) - ff' = f(x)(1 - f(x))(1 - 2f(x))$.

- **Logarithmic derivative:** $\frac{f'(x)}{f(x)} = 1 - f(x)$, so the relative rate of change of f is the complement of f itself.
- **Inverse:** for $y \in (0, 1)$, solving $y = \frac{1}{1+e^{-x}}$ for x gives

$$\begin{aligned} \frac{1}{y} &= 1 + e^{-x} \\ e^{-x} &= \frac{1 - y}{y} \\ x &= \log\left\{\frac{y}{1 - y}\right\}. \end{aligned}$$

So $f^{-1}(y) = \text{logit}\{y\}$, the log-odds. The inverse exists because $f' = f(1 - f) > 0$, so f is strictly increasing and therefore one-to-one.

- **Complement:** $1 - f(x) = \frac{e^{-x}}{1 + e^{-x}} = \frac{1}{e^x + 1} = f(-x)$.
- **Reciprocal:** $\frac{1}{f(x)} = 1 + e^{-x}$.
- **Antiderivative:** $F(x) = \log\{1 + e^x\}$ satisfies $F'(x) = \frac{e^x}{1 + e^x} = f(x)$, so $\int f(x) dx = \log\{1 + e^x\} + C$.
- **Key values:** $f(0) = \frac{1}{1+1} = \frac{1}{2}$ and $f(1) = \frac{1}{1+e^{-1}}$. As $x \rightarrow \infty$, $e^{-x} \rightarrow 0$, so $f(x) \rightarrow 1$; as $x \rightarrow -\infty$, $e^{-x} \rightarrow \infty$, so $f(x) \rightarrow 0$.

The code checks these results numerically, using the base R functions `plogis()` for f and `qlogis()` for f^{-1} , and `D()` to differentiate symbolically.

```

f <- plogis
x <- c(-2, -0.5, 0, 1, 3)

f_expr <- quote(1 / (1 + exp(-x)))
f_prime <- eval(D(f_expr, "x"), list(x = x))
f_second <- eval(D(D(f_expr, "x"), "x"), list(x = x))

checks <- c(
  derivative = max(abs(f_prime - f(x) * (1 - f(x)))),
  second_derivative = max(abs(f_second - f(x) * (1 - f(x)) * (1 - 2 * f(x)))),
  log_derivative = max(abs(f_prime / f(x) - (1 - f(x)))),
  inverse = max(abs(qlogis(f(x)) - x)),
  complement = max(abs(1 - f(x) - f(-x))),
  reciprocal = max(abs(1 / f(x) - (1 + exp(-x)))),
  antiderivative = abs(
    integrate(f, 0, 1)$value - (log(1 + exp(1)) - log(2))
  )
)
signif(checks, 3)
#>      derivative second_derivative    log_derivative      inverse
#>      1.46e-16      1.18e-16      1.53e-16      3.11e-15
#>      complement      reciprocal    antiderivative
#>      1.46e-16      0.00e+00      0.00e+00
stopifnot(all(checks < 1e-14))

key_values <- c(f(0), f(1), f(10), f(-10))

```

Each entry of `checks` is the largest discrepancy between a formula above and a numerical computation; every one is below 10^{-14} (the chunk stops with an error otherwise), which is rounding error. The key values are $f(0) = 0.5$ and $f(1) \approx 0.7311$. At $x = 10$ and $x = -10$ the function is already close to its limits, $f(10) \approx 0.99995$ and $f(-10) \approx 0.0000454$. Figure 1 graphs f , f' and $\log\{f\}$.

```

grid <- seq(-6, 6, length.out = 241)
panels <- data.frame(
  x = rep(grid, 3),
  panel = factor(
    rep(c("f(x)", "f'(x)", "log f(x)"), each = length(grid)),
    levels = c("f(x)", "f'(x)", "log f(x)")
  ),
  value = c(f(grid), f(grid) * (1 - f(grid)), log(f(grid)))
)
ggplot2::ggplot(panels) +
  ggplot2::aes(x = x, y = value) +
  ggplot2::geom_line() +
  ggplot2::facet_wrap(~panel, scales = "free_y") +
  ggplot2::labs(x = "Input, x", y = NULL)

```

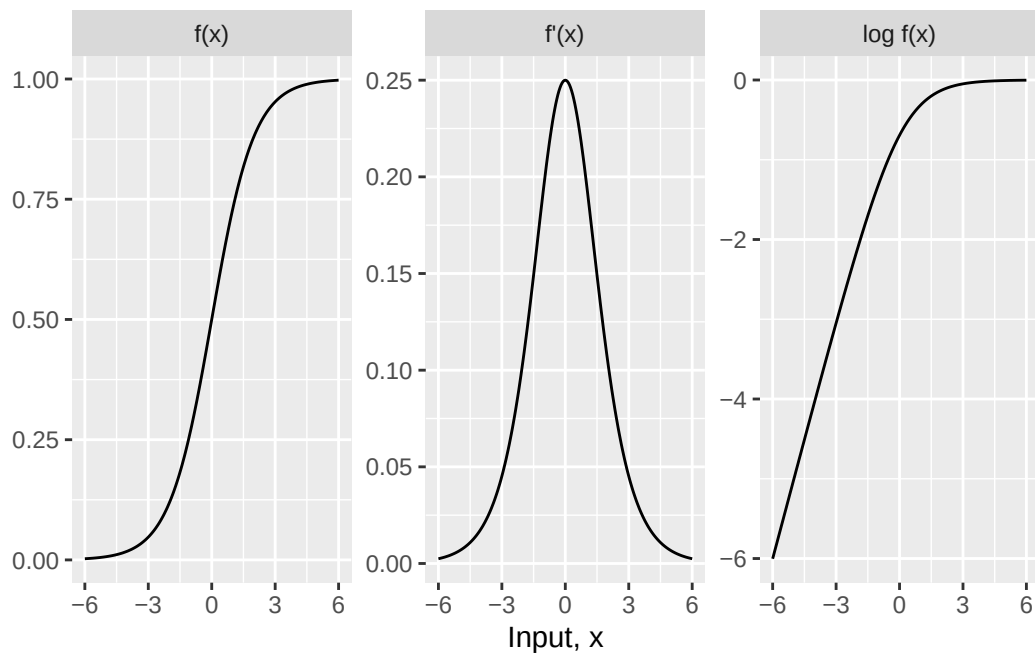


Figure 1: The logistic function, its derivative and its logarithm

Tangent lines

Definition 0.16 (Tangent line). Let f be differentiable at c (Definition 0.6). The **tangent line** to the graph of f at c is the graph of the **affine function**

$$x \mapsto f(c) + f'(c)(x - c),$$

the line through the point $(c, f(c))$ with **slope** $f'(c)$. The slope $f'(c)$ is also called the **tangent slope** of f at c .

Example 0.9 (The tangent line to x^2 at 1). For $f(x) = x^2$ at $c = 1$, $f(1) = 1$ and $f'(1) = 2$ (Example 0.5), so the tangent line is the graph of

$$\begin{aligned} x &\mapsto 1 + 2(x - 1) && \text{(substitute } f(1) = 1 \text{ and } f'(1) = 2) \\ &= 2x - 1 && \text{(distribute, and } 1 - 2 = -1) \end{aligned}$$

At $x = 1.1$ the tangent line has height $2(1.1) - 1 = 1.2$, close to the curve's height $f(1.1) = 1.21$; the gap, $1.21 - 1.2 = 0.01$, is $(1.1 - 1)^2$.

Linear approximation

For a differentiable function f and a small step ε ,

$$f(w + \varepsilon) \approx f(w) + \varepsilon \frac{d}{dw} f(w) \tag{1}$$

Definition 0.17 (Linear approximation (first-order Taylor approximation)). For a function f that is differentiable at w , the **linear approximation** of f at w [14] (also called the **first-order Taylor approximation**) is the function of the step ε

$$\hat{f}_w(\varepsilon) = f(w) + \varepsilon \frac{d}{dw} f(w)$$

so Equation 1 says $f(w + \varepsilon) \approx \hat{f}_w(\varepsilon)$ for small ε . Figure 2 illustrates this approximation interactively for a quadratic function.

i Remark (The “linear” approximation is affine)

Remark 0.2 (The “linear” approximation is affine). The linear approximation $\hat{f}_w$ (Definition 0.17) is an **affine function** of the step ε : its slope is $\frac{d}{dw} f(w)$ and its intercept is $f(w)$.

It is [linear](#) in ε only when $f(w) = 0$. The name “linear approximation” uses “linear” in the looser sense of elementary algebra ([remark](#)). Boyd and Vandenberghe [15], section 2.2, treats the first-order Taylor approximation as an affine function for functions of several variables as well.

(This plot is interactive; see the [HTML](#) or [slide version](#) of this page to view it.)

Figure 2: The linear approximation Equation 1 for $f(w) = w^2 - 4w + 7$, at any w and step ε .

Exercise 0.13 (Tangent line approximation). Adapted from Miller [10], Question 1.1.8. Let $f(x) = 4x + \sqrt{2}\cos x$. Find $f'(x)$.

Solution

Solution 0.13. Differentiate each [term](#):

$$f'(x) = 4 - \sqrt{2}\sin x$$

Exercise 0.14 (Tangent line at a point). In the setting of Exercise [0.13](#), find the equation of the [tangent line](#) to the curve $y = f(x)$ at $x = \pi/4$.

Solution

Solution 0.14. Evaluate f and f' at $x_0 = \pi/4$:

$$\begin{aligned} f(\pi/4) &= 4\left(\frac{\pi}{4}\right) + \sqrt{2}\cos\left(\frac{\pi}{4}\right) \\ &= \pi + \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) \\ &= \pi + 1 \\ f'(\pi/4) &= 4 - \sqrt{2}\sin\left(\frac{\pi}{4}\right) \\ &= 4 - \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

Using the point-slope formula, the tangent line at $(\pi/4, \pi + 1)$ is:

$$y - (\pi + 1) = 3\left(x - \frac{\pi}{4}\right) \implies y = \pi + 1 + 3\left(x - \frac{\pi}{4}\right)$$

Exercise 0.15 (Approximate with the tangent line). Use the [tangent line](#) from Exercise 0.14 to approximate $f(\pi/4 + 0.01)$.

Solution

Solution 0.15. At $x = \pi/4 + 0.01$, the [linear approximation](#) (Equation 1) gives:

$$f(\pi/4 + 0.01) \approx (\pi + 1) + 3(0.01) = \pi + 1.03 \approx 4.171593$$

Exercise 0.16 (Check the approximation). Compare the approximation in Exercise 0.15 with the exact [value](#) of $f(\pi/4 + 0.01)$.

Solution

Solution 0.16. The exact [value](#) is:

$$f(\pi/4 + 0.01) = 4\left(\frac{\pi}{4} + 0.01\right) + \sqrt{2}\cos\left(\frac{\pi}{4} + 0.01\right) \approx 4.171543$$

The approximation error is $|4.171593 - 4.171543| \approx 0.00005$, on the order of $(0.01)^2 = 10^{-4}$.

Taylor series

Definition 0.18 (Taylor polynomial). Let f be a function that has at least n derivatives at a point x_0 . The **Taylor polynomial** of degree n for f centered at x_0 is:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned} \tag{2}$$

Example 0.10 (Taylor series of the exponential function). For $f(x) = e^x$ centered at $x_0 = 0$, every derivative is

$$\begin{aligned} f^{(k)}(0) &= e^0 \\ &= 1. \end{aligned}$$

The degree- n Maclaurin polynomial is:

$$\begin{aligned} P_n(x) &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \\ &= \sum_{k=0}^n \frac{x^k}{k!} \end{aligned}$$

Because the remainder $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in \mathbb{R}$, the Taylor series converges everywhere to e^x :

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Definition 0.19 (Maclaurin polynomial). When $x_0 = 0$, the Taylor polynomial $P_n(x)$ (Definition 0.18) is called the **Maclaurin polynomial**.

Definition 0.20 (Taylor series). Let f and x_0 be as in Definition 0.18. When f is infinitely differentiable and the series converges to $f(x)$ on an open interval containing x_0 , the infinite sum $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$ is the **Taylor series** of f centered at x_0 .

Video

Chapter 11 of the *Essence of calculus* series by 3Blue1Brown develops the geometric and polynomial intuition behind Taylor approximations:

- [Taylor series](#) shows how higher-order derivatives match the curvature and rate of change of a function near a point to build polynomial approximations (Definition 0.18).

Exercise 0.17 (Taylor polynomial of a polynomial at the origin). Adapted from Miller [10], Question 1.1.16.

Find the first five terms (up to degree 4) of the Taylor series for

$$f(x) = x^8 + x^4 + 3$$

at $x_0 = 0$.

Solution

Solution 0.17. Evaluate f and its first four derivatives at $x = 0$:

$$\begin{aligned}f(x) &= x^8 + x^4 + 3 && \implies f(0) = 3 \\f'(x) &= 8x^7 + 4x^3 && \implies f'(0) = 0 \\f''(x) &= 56x^6 + 12x^2 && \implies f''(0) = 0 \\f'''(x) &= 336x^5 + 24x && \implies f'''(0) = 0 \\f^{(4)}(x) &= 1680x^4 + 24 && \implies f^{(4)}(0) = 24\end{aligned}$$

By Equation 2:

$$\begin{aligned}P_4(x) &= 3 + 0x + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 \\&= 3 + x^4\end{aligned}$$

Because f is already a polynomial centered at 0, its Taylor polynomial of degree 4 simply collects its terms of degree at most 4.

Exercise 0.18 (Taylor expansion of a polynomial at a nonzero point). Adapted from Miller [10], Question 1.1.17.

Find the first three terms (up to degree 2) of the Taylor series for

$$f(x) = x^8 + x^4 + 3$$

at $x_0 = 1$.

Solution

Solution 0.18. Evaluate f and its first two derivatives at $x_0 = 1$:

$$\begin{aligned}f(1) &= 1^8 + 1^4 + 3 \\&= 5 \\f'(1) &= 8(1)^7 + 4(1)^3 \\&= 12 \\f''(1) &= 56(1)^6 + 12(1)^2 \\&= 68\end{aligned}$$

The degree-2 Taylor polynomial is:

$$\begin{aligned}
P_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\
&= 5 + 12(x-1) + 34(x-1)^2
\end{aligned}$$

Example 0.11 (Taylor series of the cosine function). For $f(x) = \cos x$ centered at $x_0 = 0$, the derivatives follow a repeating cycle of length 4:

$$\begin{aligned}
f(0) &= \cos(0) \\
&= 1, \\
f'(0) &= -\sin(0) \\
&= 0, \\
f''(0) &= -\cos(0) \\
&= -1, \\
f'''(0) &= \sin(0) \\
&= 0,
\end{aligned}$$

and in general $f^{(2k)}(0) = (-1)^k$ and $f^{(2k+1)}(0) = 0$ for all $k \geq 0$. The Taylor series (Maclaurin series) converges everywhere to $\cos x$:

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}
\end{aligned}$$

Example 0.12 (Taylor series of the sine function). For $f(x) = \sin x$ centered at $x_0 = 0$, the derivatives evaluate at 0 to $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$ for all $k \geq 0$. The Taylor series converges everywhere to $\sin x$:

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}
\end{aligned}$$

Exercise 0.19 (Taylor expansion of a scaled cosine). Adapted from Miller [10], Question 1.1.18.

Find the terms up to degree 4 in the Taylor series for

$$f(x) = \cos(5x)$$

at $x_0 = 0$.

Solution

Solution 0.19. Recall the standard Maclaurin series for cosine:

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \dots$$

Substitute $u = 5x$:

$$\begin{aligned}\cos(5x) &= 1 - \frac{(5x)^2}{2!} + \frac{(5x)^4}{4!} - \dots \\ &= 1 - \frac{25}{2}x^2 + \frac{625}{24}x^4 - \dots\end{aligned}$$

Remark: Substituting into known Taylor expansions avoids computing high-order derivatives by hand.

Exercise 0.20 (Taylor series of a power of a function). Adapted from Miller [10], Question 1.1.19.

Find the terms up to degree 4 in the Taylor series for

$$f(x) = \cos^3(5x)$$

at $x_0 = 0$.

Solution

Solution 0.20. From Exercise 0.19, the expansion of $\cos(5x)$ up to order x^4 is:

$$\cos(5x) = 1 - \frac{25}{2}x^2 + \frac{625}{24}x^4 + O(x^6)$$

Cube this expansion using $(1 - y)^3 = 1 - 3y + 3y^2 - y^3$ with $y = \frac{25}{2}x^2 - \frac{625}{24}x^4$:

$$\begin{aligned}\cos^3(5x) &= 1 - 3\left(\frac{25}{2}x^2 - \frac{625}{24}x^4\right) + 3\left(\frac{25}{2}x^2\right)^2 + O(x^6) \\ &= 1 - \frac{75}{2}x^2 + \frac{625}{8}x^4 + \frac{1875}{4}x^4 + O(x^6) \\ &= 1 - \frac{75}{2}x^2 + \frac{4375}{8}x^4 + O(x^6)\end{aligned}$$

Exercise 0.21 (Linear Taylor polynomial of the exponential). Adapted from Miller [10], Question 1.1.20.

Find the first two terms (degree 1) of the Taylor series for

$$f(x) = e^x$$

at $x_0 = 0$.

Solution

Solution 0.21. Since

$$\begin{aligned} f(0) &= e^0 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f'(0) &= e^0 \\ &= 1 : \end{aligned}$$

$$\begin{aligned} P_1(x) &= f(0) + f'(0)x \\ &= 1 + x \end{aligned}$$

The full Taylor series is

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \end{aligned}$$

Exercise 0.22 (Taylor series of a composite exponential). Adapted from Miller [10], Question 1.1.21.

Find the first six terms (up to degree 5) of the Taylor series for

$$f(x) = \exp\{x^8\}$$

at $x_0 = 0$.

Solution

Solution 0.22. Substitute $u = x^8$ into $e^u = 1 + u + \frac{u^2}{2!} + \dots$:

$$\exp\{x^8\} = 1 + x^8 + \frac{x^{16}}{2} + \dots$$

Because the lowest-order variable term is x^8 , all coefficients for powers x^1, x^2, x^3, x^4, x^5 are zero:

$$\begin{aligned} P_5(x) &= 1 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5 \\ &= 1 \end{aligned}$$

Exercise 0.23 (Taylor expansion of the normal density). Adapted from Miller [10], Question 1.1.22.

Find the terms up to degree 3 of the Taylor series for the standard normal density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$$

at $x_0 = 0$.

Solution

Solution 0.23. Using $e^u = 1 + u + O(u^2)$ with $u = -x^2/2$:

$$\exp\left\{-\frac{x^2}{2}\right\} = 1 - \frac{x^2}{2} + O(x^4)$$

Multiplying by the normalization constant $1/\sqrt{2\pi}$:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} - \frac{x^2}{2\sqrt{2\pi}} + O(x^4)$$

Up to degree 3, the odd powers vanish by symmetry, so:

$$P_3(x) = \frac{1}{\sqrt{2\pi}} - \frac{x^2}{2\sqrt{2\pi}}$$

Theorem 0.13 (Maclaurin polynomial of the standard normal density). *The Maclaurin polynomial (Definition 0.19) of degree 3 of the standard normal density $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$ is*

$$P_3(x) = \frac{1}{\sqrt{2\pi}} - \frac{x^2}{2\sqrt{2\pi}}$$

i Proof

Proof. This is the solution to Exercise 0.23, worked in Solution 0.23. □

Exercise 0.24 (Taylor expansion of the square root). Adapted from Miller [10], Question 1.1.23.

Find the first three terms (up to degree 2) of the Taylor series for

$$f(x) = \sqrt{x}$$

at $x_0 = 1/3$.

Solution

Solution 0.24. Compute the values at $x_0 = 1/3$:

$$\begin{aligned} f(1/3) &= \left(\frac{1}{3}\right)^{1/2} \\ &= \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \\ f'(x) &= \frac{1}{2}x^{-1/2} \\ \Rightarrow f'(1/3) &= \frac{1}{2}\sqrt{3} \\ &= \frac{\sqrt{3}}{2} \\ f''(x) &= -\frac{1}{4}x^{-3/2} \\ \Rightarrow f''(1/3) &= -\frac{1}{4} \cdot 3\sqrt{3} \\ &= -\frac{3\sqrt{3}}{4} \end{aligned}$$

The degree-2 Taylor polynomial is:

$$P_2(x) = \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\left(x - \frac{1}{3}\right) - \frac{3\sqrt{3}}{8}\left(x - \frac{1}{3}\right)^2$$

Exercise 0.25 (Taylor expansion of a fractional power). Adapted from Miller [10], Question 1.1.24.

Find the first three terms of the Taylor series for

$$f(x) = (1+x)^{1/3}$$

at $x_0 = 1/2$.

Solution

Solution 0.25. At $x_0 = 1/2$, $1+x_0 = 3/2$:

$$\begin{aligned} f(1/2) &= \left(\frac{3}{2}\right)^{1/3} \\ f'(x) &= \frac{1}{3}(1+x)^{-2/3} \\ \Rightarrow f'(1/2) &= \frac{1}{3}\left(\frac{3}{2}\right)^{-2/3} \\ &= \frac{1}{3}\left(\frac{2}{3}\right)^{2/3} \\ f''(x) &= -\frac{2}{9}(1+x)^{-5/3} \\ \Rightarrow f''(1/2) &= -\frac{2}{9}\left(\frac{3}{2}\right)^{-5/3} \\ &= -\frac{2}{9}\left(\frac{2}{3}\right)^{5/3} \end{aligned}$$

Dividing $f''(1/2)$ by $2!$ gives $-\frac{1}{9}\left(\frac{2}{3}\right)^{5/3}$. The Taylor polynomial is:

$$P_2(x) = \left(\frac{3}{2}\right)^{1/3} + \frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\left(x - \frac{1}{2}\right) - \frac{1}{9}\left(\frac{2}{3}\right)^{5/3}\left(x - \frac{1}{2}\right)^2$$

Exercise 0.26 (Taylor expansion of $x \log x$). Adapted from Miller [10], Question 1.1.25. Find the first three terms (up to degree 2) of the Taylor series for

$$f(x) = x \log x$$

at $x_0 = 1$.

Solution

Solution 0.26. Evaluate derivatives at $x_0 = 1$:

$$\begin{aligned}f(1) &= 1 \log 1 \\&= 0 \\f'(x) &= \log x + 1 \\ \implies f'(1) &= 0 + 1 \\&= 1 \\f''(x) &= \frac{1}{x} \implies f''(1) = 1\end{aligned}$$

Therefore:

$$\begin{aligned}P_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\&= (x-1) + \frac{1}{2}(x-1)^2\end{aligned}$$

Exercise 0.27 (Maclaurin series of $\log(1+x)$). Adapted from Miller [10], Question 1.1.26. Find the first three terms of the Taylor series for

$$f(x) = \log(1+x)$$

at $x_0 = 0$.

Solution

Solution 0.27. Differentiating repeatedly at $x_0 = 0$:

$$\begin{aligned}f(0) &= \log 1 \\&= 0 \\f'(x) &= (1+x)^{-1} \implies f'(0) = 1 \\f''(x) &= -(1+x)^{-2} \implies f''(0) = -1 \\f'''(x) &= 2(1+x)^{-3} \implies f'''(0) = 2\end{aligned}$$

The Taylor polynomial of degree 3 is:

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Theorem 0.14 (Maclaurin polynomial of $\log(1+x)$). *The Maclaurin polynomial (Definition 0.19) of degree 3 of $f(x) = \log(1+x)$ is*

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

i Proof

Proof. This is the solution to Exercise 0.27, worked in Solution 0.27. □

Exercise 0.28 (Maclaurin series of $\log(1-x)$). Adapted from Miller [10], Question 1.1.27. Find the terms up to degree 3 of the Taylor series for

$$f(x) = \log(1-x)$$

at $x_0 = 0$.

Solution

Solution 0.28. Substitute $u = -x$ into the expansion for $\log(1+u)$ from Theorem 0.14:

$$\begin{aligned}\log(1-x) &= (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \dots \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\end{aligned}$$

Theorem 0.15 (Maclaurin polynomial of $\log(1-x)$). *The Maclaurin polynomial (Definition 0.19) of degree 3 of $f(x) = \log(1-x)$ is*

$$P_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

i Proof

Proof. This is the solution to Exercise 0.28, worked in Solution 0.28. □

Exercise 0.29 (Taylor series of $\log((1-x)e^x)$). Adapted from Miller [10], Question 1.1.28.

Find the first two non-zero terms of the Taylor series for

$$f(x) = \log((1-x)e^x)$$

at $x_0 = 0$.

Solution

Solution 0.29. Expand using logarithm properties:

$$\begin{aligned} f(x) &= \log(1-x) + \log(e^x) \\ &= \log(1-x) + x \end{aligned}$$

Substitute the series for $\log(1-x)$ from Theorem 0.15:

$$\begin{aligned} f(x) &= \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right) + x \\ &= -\frac{x^2}{2} - \frac{x^3}{3} - \dots \end{aligned}$$

The constant and linear terms vanish ($f(0) = 0$, $f'(0) = 0$). The first two non-zero terms are:

$$-\frac{x^2}{2} - \frac{x^3}{3}$$

Exercise 0.30 (Taylor series of a product of known expansions). Adapted from Miller [10], Question 1.1.29.

Find the terms up to degree 2 of the Taylor series for

$$f(x) = \cos(x) \log(1+x)$$

at $x_0 = 0$.

Solution

Solution 0.30. Multiply the known Maclaurin expansions:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + O(x^4) \\ \log(1+x) &= x - \frac{x^2}{2} + O(x^3) \end{aligned}$$

Multiplying and dropping terms of order x^3 and higher:

$$\begin{aligned} f(x) &= \left(1 - \frac{x^2}{2} + \dots\right) \left(x - \frac{x^2}{2} + \dots\right) \\ &= x - \frac{x^2}{2} + O(x^3) \end{aligned}$$

Exercise 0.31 (Taylor series of $\log(1 + 2x)$). Adapted from Miller [10], Question 1.1.30. Find the first two non-zero terms of the Taylor series for

$$f(x) = \log(1 + 2x)$$

at $x_0 = 0$.

Solution

Solution 0.31. Substitute $u = 2x$ into the Maclaurin series $\log(1 + u) = u - \frac{u^2}{2} + \dots$:

$$\begin{aligned} \log(1 + 2x) &= (2x) - \frac{(2x)^2}{2} + \dots \\ &= 2x - 2x^2 + \dots \end{aligned}$$

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