

Trigonometry

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Cosine and sine

Definition 0.1 (Unit circle). The **unit circle** [1] is the set of points (x_1, x_2) in the plane with $x_1^2 + x_2^2 = 1$.

Example 0.1 (Points on the unit circle).

- $(1, 0)$ is on the unit circle, because $1^2 + 0^2 = 1$.
- $(0, -1)$ is on the unit circle, because $0^2 + (-1)^2 = 1$.
- $(\frac{3}{5}, \frac{4}{5})$ is on the unit circle, because $\frac{9}{25} + \frac{16}{25} = 1$.
- $(1, 1)$ is not on the unit circle, because $1^2 + 1^2 = 2 \neq 1$.

Definition 0.2 (The number π). The number $\pi \approx 3.14159$ [2] is half of the circumference of the unit circle (Definition 0.1).

Example 0.2 (Distances around the unit circle). The whole unit circle has circumference 2π . Moving from $(1, 0)$ to $(-1, 0)$ along the upper half of the circle covers a distance of π , and moving from $(1, 0)$ to $(0, 1)$ covers a quarter of the circle, a distance of $\pi/2$.

Definition 0.3 (Angle on the unit circle). The **angle** between two points of the **unit circle** is how far apart they are along the circle, going the short way round. Let the counterclockwise distance from the first point to the second be $s \in [0, 2\pi)$. Then the angle is

$$\theta \stackrel{\text{def}}{=} \min\{s, 2\pi - s\}.$$

In words, the angle is the smaller of two distances from the first point to the second: the counterclockwise one, s , or the clockwise one, $2\pi - s$.

Example 0.3 (Three angles on the unit circle).

- The points $(1, 0)$, $(0, 1)$ are a quarter of the circle apart, so their **angle** is $\pi/2$.
- The points $(1, 0)$, $(-1, 0)$ are half of the circle apart, so their angle is π .
- The point $(1, 0)$ is 0 apart from itself, so their angle is 0.

Definition 0.4 (Radian). A **radian** is the **angle on the unit circle** between two points that are a distance of 1 apart along the circle. An **angle** of θ is then θ radians.

Example 0.4 (Angles in radians).

- The points $(0, 1)$, $(-1, 0)$ are a quarter of the **unit circle** apart, so their **angle** is $\pi/2$ **radians**.
- The points $(0, 1)$, $(0, -1)$ are half of the circle apart, so their angle is π radians.

Proposition 0.1 (Range of an angle on the unit circle). An **angle** between two points of the **unit circle** is at least 0, at most π (Definition 0.2).

i Proof

Proof. The whole circle has length 2π (Definition 0.2). Going the short way round covers at most half of the circle, which is π . A distance along the circle is at least 0. $\square$

Definition 0.5 (Cosine). For a real number θ , start at the point $(1, 0)$ of the unit circle (Definition 0.1) and move a distance $|\theta|$ along the circle, counterclockwise if $\theta \geq 0$ and clockwise if $\theta < 0$. The first coordinate of the point reached is the **cosine** of θ [3], written $\cos \theta$.

Example 0.5 (Cosines of some angles).

- $\theta = 0$ stays at $(1, 0)$, so $\cos 0 = 1$.
- $\theta = \pi$ goes half way around the circle, to $(-1, 0)$, so $\cos \pi = -1$.
- $\theta = \pi/2$ goes a quarter of the way round, to $(0, 1)$, so $\cos(\pi/2) = 0$.
- $\theta = \pi/4$ goes an eighth of the way round, to the point of the circle with $x_1 = x_2 > 0$. There $x_1^2 + x_2^2 = 1$, so $x_1^2 = \frac{1}{2}$ and $\cos(\pi/4) = x_1 = \frac{1}{\sqrt{2}} \approx 0.7071$.

Angles are also measured in *degrees*, with a full turn of 2π equal to 360 degrees, so $\pi/2$ is 90 degrees and $\pi/4$ is 45 degrees.

Definition 0.6 (Sine). For a real number θ , the second coordinate of the point reached from $(1, 0)$ by moving a distance $|\theta|$ along the unit circle (Definition 0.5) is the **sine** of θ [3], written $\sin \theta$. The sine is one of the trigonometric functions [4].

Example 0.6 (Sines of some angles).

- $\theta = 0$ stays at $(1, 0)$, so $\sin 0 = 0$.
- $\theta = \pi$ goes half way around the circle, to $(-1, 0)$, so $\sin \pi = 0$.
- $\theta = \pi/2$ goes a quarter of the way round, to $(0, 1)$, so $\sin(\pi/2) = 1$.
- $\theta = \pi/4$ reaches the point with $x_1 = x_2 = \frac{1}{\sqrt{2}}$ (Example 0.5), so $\sin(\pi/4) = \frac{1}{\sqrt{2}}$.

Theorem 0.1 (The Pythagorean identity). *For every real number θ [5],*

$$\cos^2 \theta + \sin^2 \theta = 1.$$

i Proof

Proof. The point $(\cos \theta, \sin \theta)$ lies on the unit circle (Definition 0.1), so its coordinates satisfy $x_1^2 + x_2^2 = 1$. □

Example 0.7 (The unit circle and the angle $\pi/3$). Take $\theta = \pi/3$, one sixth of the way round the circle. Figure 1 shows the point reached, $(\cos \theta, \sin \theta)$, and its two coordinates. The point is a vertex of a regular hexagon inscribed in the unit circle, whose sides have the same length as its radius, so the point is at distance 1 from $(1, 0)$. Writing $(x_1, x_2) = (\cos \theta, \sin \theta)$,

$$\begin{aligned} 1 &= (x_1 - 1)^2 + x_2^2 \\ &= x_1^2 + x_2^2 - 2x_1 + 1 \\ &= 2 - 2x_1, \end{aligned}$$

so $\cos(\pi/3) = x_1 = \frac{1}{2}$, and then $\sin(\pi/3) = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$. To three decimal places these are 0.5 and 0.866. As Theorem 0.1 says, the squared coordinates add to 1.

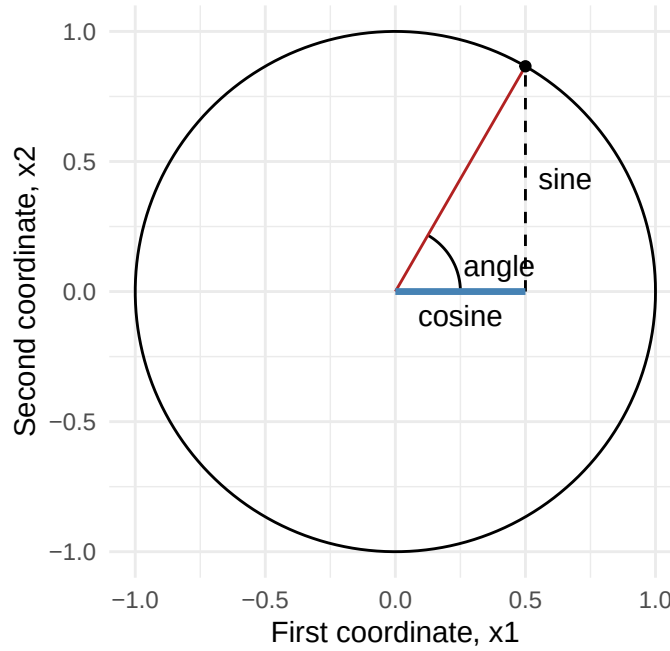


Figure 1: The unit circle with the point at angle $\pi/3$. Its horizontal coordinate is the cosine and its vertical coordinate is the sine.

Theorem 0.2 (Cosine takes each value from -1 to 1 exactly once between 0 and π). *The function $\cos : [0, \pi] \rightarrow [-1, 1]$ is decreasing, with $\cos 0 = 1$ and $\cos \pi = -1$, and takes each value in $[-1, 1]$ exactly once. On $[0, \pi]$, $\sin \theta \geq 0$.*

i Proof

Proof. For $\theta \in [0, \pi]$, the point $(\cos \theta, \sin \theta)$ is at distance θ along the circle from $(1, 0)$, counterclockwise, so it lies on the upper half of the unit circle, $\{(x_1, x_2) : x_1^2 + x_2^2 = 1, x_2 \geq 0\}$, which gives $\sin \theta \geq 0$. Different values of θ in $[0, \pi]$ are different distances along that half circle, so they give different points. Two different points of the upper half have different first coordinates, because $x_2 = \sqrt{1 - x_1^2}$ is determined by x_1 . So $\cos$ is one-to-one on $[0, \pi]$. Every $x_1 \in [-1, 1]$ is the first coordinate of the point $(x_1, \sqrt{1 - x_1^2})$ of the upper half, and the distance along the half circle from $(1, 0)$ to that point is some $\theta \in [0, \pi]$, so $\cos$ takes every value in $[-1, 1]$. Finally, move along the upper half circle from $(1, 0)$ to $(-1, 0)$ as θ increases from 0 to π . The first coordinate never takes the same value twice, so it never turns back. It starts at $\cos 0 = 1$ and ends at $\cos \pi = -1$, so it only decreases. $\square$

Definition 0.7 (Arccosine). The **arccosine** $\arccos : [-1, 1] \rightarrow [0, \pi]$ is the **function** that sends each **cosine**, $c \in [-1, 1]$, to the one **real number** $\theta \in [0, \pi]$ whose cosine is c . By Theorem 0.2, there is just one such θ .

Example 0.8 (Three arccosines).

- The **arccosine** of one is $\arccos 1 = 0$, because the **cosine** of zero is $\cos 0 = 1$.
- The arccosine of zero is $\arccos 0 = \pi/2$, because the cosine of a quarter turn is $\cos(\pi/2) = 0$.
- The arccosine of minus one is $\arccos(-1) = \pi$, because the cosine of a half turn is $\cos \pi = -1$.

- [1] Wikipedia contributors, “Unit circle — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Unit_circle
- [2] Wikipedia contributors, “Pi — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Pi>
- [3] Wikipedia contributors, “Sine and cosine — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Sine_and_cosine
- [4] Wikipedia contributors, “Trigonometric functions — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Trigonometric_functions
- [5] Wikipedia contributors, “Pythagorean trigonometric identity — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Pythagorean_trigonometric_identity