Polynomials, Affine Functions and Series

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Polynomials

Exercise 0.1 (Reading a polynomial)**.** Let $f(x) = 3x^4 - x + 7$.

1. Write f in the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, listing every coefficient a_4, a_3, a_2, a_1, a_0.
2. What is the degree of f, and what is its leading coefficient?

Solution

Solution 0.1.

1. $f(x) = 3x^4 + 0 \cdot x^3 + 0 \cdot x^2 + (-1) \cdot x + 7$, so $a_4 = 3$, $a_3 = 0$, $a_2 = 0$, $a_1 = -1$, and $a_0 = 7$. Powers of x that do not appear have coefficient 0.
2. The highest power with a nonzero coefficient is x^4, so the degree is 4 and the leading coefficient is $a_4 = 3$.

Definition 0.1 (Constant function)**.** A function $f : A \to B$ is **constant** if there is some $c \in B$ with $f(x) = c$ for all $x \in A$.

Example 0.1 (Constant and non-constant functions)**.**

- $f(x) = 7$ on $\mathbb{R}$ is constant, with $c = 7$.
- $f(x) = x^2$ on $\mathbb{R}$ is not constant: $f(0) = 0$ but $f(1) = 1$.

Definition 0.2 (Polynomial)**.** A **polynomial** (in one variable x) [1] is a function $f : \mathbb{R} \to \mathbb{R}$ that can be written as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some integer $n \geq 0$ and constants $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. The terms of f are $a_n x^n, \ldots, a_1 x, a_0$.

ℹ Remark (Constant and zero polynomials)

Remark 0.1 (Constant and zero polynomials). A constant function $f(x) = 7$ is a polynomial with $n = 0$ and $a_0 = 7$. The requirement $a_n \neq 0$ means this definition covers nonzero polynomials only: the zero function $f(x) = 0$ is excluded here, because it has no nonzero coefficient to serve as a_n.

Definition 0.3 (Coefficients of a polynomial)**.** The constants $a_0, a_1, \ldots, a_n$ of a polynomial f (Definition 0.2) are the **coefficients** of f, with a_k the coefficient of x^k.

Definition 0.4 (Degree of a polynomial)**.** Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with $a_n \neq 0$. The integer n is the **degree** of f.

Example 0.2 (Degrees of some polynomials)**.**

- $f(x) = 4x$ has degree 1.
- $f(x) = 5 - 2x^2 + x^3$ has degree 3: the degree is the highest power with a nonzero coefficient, not the power in the first term written.
- A constant polynomial $f(x) = 7$ has degree 0.

Definition 0.5 (Leading coefficient)**.** Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree n. The constant a_n is the **leading coefficient** of f.

Example 0.3 (Leading coefficients)**.**

- $f(x) = 5 - 2x^2 + x^3$ has degree 3, so its leading coefficient is $a_3 = 1$, not the 5 written first.
- $f(x) = 3 - x^2$ has leading coefficient $a_2 = -1$.

Definition 0.6 (Quadratic polynomial)**.** A polynomial of degree 2, $f(x) = a_2x^2+a_1x+a_0$ with $a_2 \neq 0$, is a **quadratic** [2] polynomial.

Definition 0.7 (Cubic polynomial)**.** A polynomial of degree 3 is a **cubic** polynomial.

Example 0.4 (Quadratic and cubic polynomials)**.**

- $$\begin{aligned} f(x) &= (x-2)^2 \\ &= x^2 - 4x + 4 \end{aligned}$$

  is quadratic, by Remark 1 in Sums, Products and Quotients with $a = x$ and $b = 2$.
- $f(x) = x^3 - 3x$ is cubic.
- $f(x) = 4x + 1$ is neither: it has degree 1.

Definition 0.8 (Parabola)**.** A **parabola** [3] is the graph of a quadratic polynomial (Definition 0.6).

Example 0.5 (The parabola $y = x^2$)**.** The graph of $f(x) = x^2$ is a parabola. It contains the points $(-1, 1)$, $(0, 0)$ and $(2, 4)$, and its lowest point is $(0, 0)$, since $x^2 \geq 0$ for every x.

Definition 0.9 (Monomial (power product))**.** A **monomial**, also called a **power product** [4], in the variables $x_1, \ldots, x_p$ is an expression

$$x_1^{a_1} x_2^{a_2} \cdots x_p^{a_p} \tag{1}$$

that multiplies powers of those variables, whose exponents $a_1, \ldots, a_p$ are nonnegative integers.
Each $a_j = 0$ contributes the factor $x_j^0 = 1$ (zero exponent).

Example 0.6 (Which expressions are monomials)**.** Take the four variables x_1, x_2, x_3, x_4.

- $x_1^3 x_4$ is a monomial, with exponents $(a_1, a_2, a_3, a_4) = (3, 0, 0, 1)$.
- The constant 1 is the monomial whose exponents are all 0.
- $x_1/x_2 = x_1 x_2^{-1}$ is not a monomial, because the exponent of x_2 is negative.
- $\sqrt{x_3} = x_3^{1/2}$ is not a monomial, because its exponent is not an integer.

- $x_1 + x_4$ is not a monomial: it adds two monomials rather than multiplying powers.

Definition 0.10 (Degree of a monomial)**.** The **degree** of the monomial $x_1^{a_1} x_2^{a_2} \cdots x_p^{a_p}$ is the sum of its exponents, $a_1 + a_2 + \cdots + a_p$ [4].

Example 0.7 (Degrees of some monomials)**.**

- $x_1^3 x_4$ has degree $3 + 1 = 4$.
- $x_2^2 x_3^5$ has degree $2 + 5 = 7$.
- The constant monomial 1 has degree 0.
- With one variable, the monomial x^n has degree n, which is the degree of the one-term polynomial $f(x) = x^n$.

Affine and linear functions of one variable

Definition 0.1 (Affine function of one variable)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **affine** [5] if there are numbers m and b such that

$$f(x) = mx + b \quad \text{for all } x \in \mathbb{R}$$

Definition 0.2 (Slope)**.** In the formula $f(x) = mx + b$ for an affine function f (Definition 0.1), m is called the **slope** of f [6].

Definition 0.3 (Intercept, y-intercept)**.** The **intercept** (or **y-intercept**) of a function f whose domain contains 0 is $f(0)$, the height at which the graph of f crosses the vertical axis.

Definition 0.4 (Intercept term)**.** When a function is written as terms added together, its **intercept term** is the term with no factor of the input. When every other term has a factor of the input, the intercept (Definition 0.3) of the function is this term.

Definition 0.5 (x-intercept)**.** An **x-intercept** of a function f is an input x with $f(x) = 0$, a point where the graph of f crosses the horizontal axis. A function can have no x-intercept, one, or many.

Remark (Intercepts of an affine function)

Remark 0.1 (Intercepts of an affine function). For an affine function $f(x) = mx + b$ (Definition 0.1):

- the intercept term is b,
- the intercept is $f(0) = b$,
- if $m \neq 0$, the only x-intercept is $-b/m$.

Definition 0.6 (Linear function of one variable)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **linear** [7] if it is affine (Definition 0.1) with intercept 0; that is, if there is a number m such that

$$f(x) = mx \quad \text{for all } x \in \mathbb{R}$$

Theorem 0.1 (Intercept and slope of an affine function)**.** *Let $f(x) = mx + b$ be an affine function (Definition 0.1).*

1. *The intercept is the value at zero: $f(0) = b$.*
2. *The slope is the change in f per unit change in x: for any $x_1 \neq x_2$,*
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = m$$
3. *So f determines its slope and intercept: if also $f(x) = m'x + b'$ for all x, then $m' = m$ and $b' = b$.*

Proof

Proof. **Part 1.**

$$\begin{aligned} f(0) &= m \cdot 0 + b && \text{(Definition 0.1)} \\ &= 0 + b && \text{(any number times 0 is 0)} \\ &= b + 0 && \text{(Theorem 2 in Sums, Products and Quotients)} \\ &= b && \text{(Theorem 1 in Sums, Products and Quotients)} \end{aligned}$$

Part 2. First the numerator:

$$
\begin{aligned}
f(x_2) - f(x_1) &= (mx_2 + b) - (mx_1 + b) && \text{(Definition 0.1)} \\
&= mx_2 + b - mx_1 - b && \text{(subtracting a sum subtracts each term)} \\
&= mx_2 - mx_1 + b - b && \text{(Theorem 2 in Sums, Products and Quotients, swapping } b \text{ and } -m \\
&= (mx_2 - mx_1) + (b - b) && \text{(Theorem 3 in Sums, Products and Quotients)} \\
&= (mx_2 - mx_1) + 0 && (b - b = 0) \\
&= mx_2 - mx_1 && \text{(Theorem 1 in Sums, Products and Quotients)} \\
&= mx_2 + m(-x_1) && (-(mx_1) = m(-x_1), \text{ by Theorem 8 in Equalities, Inequalities and } \\
&= m(x_2 + (-x_1)) && \text{(Theorem 8 in Sums, Products and Quotients, read right to left)} \\
&= m(x_2 - x_1) && \text{(adding a negative is subtracting)}
\end{aligned}
$$

Then, writing $d = x_2 - x_1$, which is not 0 because $x_1 \neq x_2$,

$$
\begin{aligned}
\frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{md}{d} && \text{(the numerator above)} \\
&= (md) \cdot \frac{1}{d} && \text{(Theorem 7 in Sums, Products and Quotients)} \\
&= m \cdot \left(d \cdot \frac{1}{d} \right) && \text{(Theorem 6 in Sums, Products and Quotients)} \\
&= m \cdot \frac{d}{d} && \text{(Theorem 7 in Sums, Products and Quotients)} \\
&= m \cdot 1 && \text{(a nonzero number divided by itself is 1)} \\
&= m && \text{(Theorem 4 in Sums, Products and Quotients)}
\end{aligned}
$$

Part 3. Part 1, applied to each formula, gives

$$
\begin{aligned}
b &= f(0) \\
&= b'.
\end{aligned}
$$

Part 2 with $x_1 = 0$ and $x_2 = 1$, applied to each formula, gives

$$
\begin{aligned}
m &= f(1) - f(0) \\
&= m'.
\end{aligned}
$$

□

Example 0.1 (An affine function, and a linear one)**.** $f(x) = 2x + 3$ is affine with slope 2 and intercept 3. By Theorem 0.1, $f(0) = 3$, and each unit step in x raises f by 2: for example, $f(1) = 5$ and $f(2) = 7$. Its intercept is not 0, so it is not linear.
$g(x) = -\frac{1}{2}x$ is affine with intercept 0, so it is linear: $g(0) = 0$, and each unit step in x lowers g by $\frac{1}{2}$.

Remark (Elementary algebra calls $mx + b$ "linear")

Remark 0.2 (Elementary algebra calls $mx + b$ "linear"). Elementary algebra usually calls $f(x) = mx + b$ a *linear function*, because its graph is a straight line. This site calls that an affine function and keeps "linear" for the case $b = 0$, because that is the sense used in linear algebra, where a linear map must send 0 to 0 (theorem).

Limits of sequences

Definition 0.1 (Limit of a sequence)**.** A real number L is the **limit** of a sequence (a_n) of real numbers [8], written $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, if for every $\varepsilon > 0$ there is a natural number N such that

$$|a_n - L| < \varepsilon \quad \text{for every } n \geq N.$$

Example 0.1 (The limit of a sequence)**.** $a_n = \frac{1}{n}$ has limit 0. Given $\varepsilon > 0$, take N to be any natural number larger than $\frac{1}{\varepsilon}$. For every $n \geq N$, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. For example, with $\varepsilon = 0.01$, take $N = 101$: every $n \geq 101$ has $\frac{1}{n} \leq \frac{1}{101} < 0.01$.

Definition 0.2 (Convergent sequence (converges))**.** A sequence (a_n) of real numbers **converges** to a real number L if L is its limit (Definition 0.1). A sequence that converges to some real number is **convergent**.

Example 0.2 (A convergent sequence)**.** $a_n = \frac{1}{n}$ converges to 0, because 0 is its limit (Example 0.1), so $a_n = \frac{1}{n}$ is convergent.

Definition 0.3 (Divergent sequence (diverges))**.** A sequence that does not converge (Definition 0.2) to any real number **diverges**, and a sequence that diverges is **divergent**.

Example 0.3 (A divergent sequence)**.** $c_n = n$ diverges. For any real number L and $\varepsilon = 1$, every $n > L + 1$ has $|n - L| > 1$, so no N works. For example, with $L = 5$, every $n \geq 7$ has $|n - 5| \geq 2$. It diverges to ∞: for every real number M, every $n \geq N$ has $c_n = n > M$ when N is a natural number larger than M.

Definition 0.4 (Divergence of a sequence to infinity)**.** A sequence (a_n) of real numbers **diverges to** ∞, written $\lim_{n\to\infty} a_n = \infty$, if for every real number M there is a natural number N such that $a_n > M$ for every $n \geq N$.

Series

Definition 0.1 (Non-decreasing sequence)**.** A sequence $a_1, a_2, \ldots$ of real numbers or of extended non-negative reals is **non-decreasing** if each term is at least as large as the one before it:

$$a_n \leq a_{n+1} \quad \text{for every } n \in \mathbb{N}$$

Example 0.1 (Non-decreasing and not)**.**

- $1, 2, 2, 3, 3, 3, \ldots$ is non-decreasing: a term may equal the one before it.
- $1, 2, \infty, \infty, \ldots$ is non-decreasing, because ∞ is greater than every real number.
- $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is not non-decreasing, because

$$\begin{aligned} a_2 &= \frac{1}{2} \\ &< 1 \\ &= a_1. \end{aligned}$$

Definition 0.2.

Partial sum
Let $a_1, a_2, \ldots$ be a sequence of real numbers or of extended non-negative reals. For each $n \in \mathbb{N}$, its nth **partial sum** [9] is the sum of its first n terms:

$$s_n \stackrel{\text{def}}{=} \sum_{i=1}^{n} a_i = a_1 + \cdots + a_n$$

Example 0.2 (Partial sums of halves)**.** For $a_i = 1/2^i$, the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, the first three partial sums are $s_1 = \frac{1}{2}$,

$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

and

$$s_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}.$$

In general, $s_n = 1 - 1/2^n$.

Definition 0.3 (Infinite sum, series)**.** Let $a_1, a_2, \ldots$ be a sequence in $[0, \infty]$, with partial sums s_n (Definition 0.2). The **infinite sum** (also called a **series**) [9] $\sum_{i=1}^{\infty} a_i$ is the limit of the partial sums:

$$\sum_{i=1}^{\infty} a_i \stackrel{\text{def}}{=} \lim_{n \to \infty} s_n$$

Here the limit is a real number when the partial sums are real numbers that converge to it, and it is ∞ when the partial sums diverge to ∞ or some partial sum is ∞. Sums with other index ranges, such as $\sum_{k=0}^{\infty} a_k$, are defined the same way.

Example 0.3 (A finite and an infinite sum)**.**

- For $a_i = 1/2^i$, the partial sums $s_n = 1 - 1/2^n$ (Example 0.2) converge to 1, because $1/2^n \to 0$. So $\sum_{i=1}^{\infty} 1/2^i = 1$.
- For $a_i = 1$, the partial sums are $s_n = n$, which diverge to ∞. So $\sum_{i=1}^{\infty} 1 = \infty$.

Lemma 0.1 (Sums of non-negative terms)**.** *Let $a_1, a_2, \ldots$ be values in $[0, \infty]$, with partial sums s_n (Definition 0.2), where $x + \infty = \infty$ for every x in $[0, \infty]$. Then the partial sums are non-decreasing, $s_1 \le s_2 \le \cdots$, so their limit, the infinite sum $\sum_{i=1}^{\infty} a_i$ (Definition 0.3), always exists, and it is either a finite number or ∞.*

Proof

Proof. For each n:

$$\begin{aligned} s_{n+1} &= s_n + a_{n+1} && \text{(definition of } s_{n+1}\text{)} \\ &\geq s_n && (a_{n+1} \geq 0) \end{aligned}$$

If some partial sum s_N is ∞, then $s_n = \infty$ for every $n \geq N$, so the limit is ∞. Otherwise, the partial sums form a non-decreasing sequence of real numbers. If that sequence is bounded above, it converges to a finite number [10]. If it is not bounded above, then for every number M some s_N exceeds M, and so does every later $s_n \geq s_N$; the limit is ∞. □

[1] Wikipedia contributors, "Polynomial — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Polynomial

[2] Wikipedia contributors, "Quadratic function — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Quadratic_function

[3] Wikipedia contributors, "Parabola — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Parabola

[4] Wikipedia contributors, "Monomial — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Monomial

[5] Wikipedia contributors, "Affine transformation — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Affine_transformation

[6] Wikipedia contributors, "Slope — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Slope

[7] Wikipedia contributors, "Linear function — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Linear_function

[8] Wikipedia contributors, "Limit of a sequence — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Limit_of_a_sequence

[9] Wikipedia contributors, "Series (mathematics) — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Series_(mathematics)

[10] Wikipedia contributors, "Monotone convergence theorem — Wikipedia, the free encyclopedia." 2026. Available: https://en.wikipedia.org/wiki/Monotone_convergence_theorem