Algebra

Last modified: 2026-09-28 23:45:45 (PDT)

Mastery of Elementary Algebra (a.k.a. "College Algebra") is a prerequisite for calculus, which is in turn a prerequisite for most statistics and data science courses. Nevertheless, each year, some students are still uncomfortable with algebraic manipulations of mathematical formulas. Therefore, I include this section as a quick reference.

Equalities

Theorem 0.1 (Equalities are transitive). *If $a = b$ and $b = c$, then $a = c$*

Theorem 0.2 (Substituting equivalent expressions). *If $a = b$, then for any function $f(x)$, $f(a) = f(b)$*

Inequalities

Theorem 0.3 (Adding to both sides of an inequality). *If $a < b$, then $a + c < b + c$*

Theorem 0.4 (negating both sides of an inequality). *If $a < b$, then: $-a > -b$*

Theorem 0.5 (Multiplying both sides of an inequality by a positive number). *If $a < b$ and $c > 0$, then $ca < cb$.*

Theorem 0.6 (Negation is multiplication by -1).

$$-a = (-1) * a$$

Infimum and supremum

Definition 0.1 (Infimum (greatest lower bound)). Let $A \subseteq \mathbb{R}$ be nonempty and bounded below, meaning that some $t \in \mathbb{R}$ satisfies $t \leq a$ for all $a \in A$. The **infimum** of A, written $\inf A$, is the greatest real number t satisfying $t \leq a$ for all $a \in A$:

$$\inf A \stackrel{\text{def}}{=} \max\{t \in \mathbb{R} : \forall a \in A,\ t \leq a\}$$

If A is nonempty but not bounded below, we write $\inf A = -\infty$ by convention.

The maximum in Definition 0.1 always exists: that is the completeness (greatest-lower-bound) property of the real numbers (Rudin 1976, Definition 1.8, p. 4, and Theorem 1.19, p. 8). If the infimum belongs to A, it equals the minimum: $\inf A = \min A$.

Example 0.1 (Numerical examples of infimum).

- $\inf\{1, 2, 3\} = 1$, since 1 is the smallest element.
- $\inf(0.5, 1] = 0.5 = \min[0.5, 1]$: for intervals open below, the infimum equals the minimum of the corresponding closed-below interval, even though $0.5 \notin (0.5, 1]$. More generally, $\inf(c, b] = \min[c, b] = c$ for any $c < b$.
- $\inf\{t \geq 0 : t > 0.5\} = 0.5$, even though 0.5 itself is not in the set.
- $\inf\{-1, -2, -3, ...\} = -\infty$, because no real number is less than or equal to every element of that set.

Definition 0.2 (Supremum (least upper bound)). Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, meaning that some $t \in \mathbb{R}$ satisfies $a \leq t$ for all $a \in A$. The **supremum** of A, written $\sup A$, is the smallest real number t satisfying $a \leq t$ for all $a \in A$:

$$\sup A \stackrel{\text{def}}{=} \min\{t \in \mathbb{R} : \forall a \in A,\ a \leq t\}$$

If A is nonempty but not bounded above, we write $\sup A = +\infty$ by convention.

The minimum in Definition 0.2 always exists: that is the completeness (least-upper-bound) property of the real numbers (Rudin 1976, Definition 1.8, p. 4, and Theorem 1.19, p. 8). If the supremum belongs to A, it equals the maximum: $\sup A = \max A$.

Example 0.2 (Numerical examples of supremum).

- $\sup\{1, 2, 3\} = 3$, since 3 is the largest element.
- $\sup\{t \geq 0 : t < 0.5\} = 0.5$, even though 0.5 itself is not in the set.
- $\sup\{1, 2, 3, ...\} = +\infty$, because no real number is greater than or equal to every element of that set.

Sums

Theorem 0.7 (adding zero changes nothing).

$$a + 0 = a$$

Theorem 0.8 (Sums are symmetric).

$$a + b = b + a$$

Theorem 0.9 (Sums are associative).
When summing three or more terms, the order in which you sum them does not matter:

$$(a + b) + c = a + (b + c)$$

Products

Theorem 0.10 (Multiplying by 1 changes nothing).

$$a \times 1 = a$$

Theorem 0.11 (Products are symmetric).

$$a \times b = b \times a$$

Theorem 0.12 (Products are associative).

$$(a \times b) \times c = a \times (b \times c)$$

Division

Theorem 0.13 (Division can be written as a product). *If* $b \neq 0$*, then*

$$\frac{a}{b} = a \times \frac{1}{b}$$

Sums and products together

Theorem 0.14 (Multiplication is distributive)**.**

$$a(b + c) = ab + ac$$

Quotients

Definition 0.3 (Quotient)**.** For real numbers a and b with $b \neq 0$, the **quotient** of a by b is the result of dividing a by b:

$$\frac{a}{b}$$

A quotient is also called a *fraction*; a is its *numerator* and b its *denominator*. A quotient whose denominator measures time or population size is often called a *rate*; in epidemiology, rates typically have such a denominator.

cf. https://en.wikipedia.org/wiki/Rate_(mathematics)

Example 0.3 (A quotient)**.** The quotient of 6 by 4 is $\frac{6}{4} = 1.5$. The quotient of 6 by 0 is undefined, because Definition 0.3 requires a nonzero denominator.

Definition 0.4 (Ratios)**.** A **ratio** is a quotient in which the numerator and denominator are measured using the same unit scales.
cf. https://en.wikipedia.org/wiki/Ratio

Definition 0.5 (Proportion)**.** In statistics, a **proportion** typically means a ratio where the numerator represents a subset of the denominator.
See https://en.wikipedia.org/wiki/Population_proportion.
See also https://en.wikipedia.org/wiki/Proportion_(mathematics) for other meanings.

Definition 0.6 (Proportional)**.** Two functions $f(x)$ and $g(x)$ are **proportional** if their ratio $\frac{f(x)}{g(x)}$ does not depend on x. (cf. https://en.wikipedia.org/wiki/Proportionality_(mathematics))

Additional reference for elementary algebra: https://en.wikipedia.org/wiki/Population_proportion#Mathematical_definition

Exponentials and Logarithms

In these notes, $\log\{x\}$ is the natural logarithm of $x > 0$, the logarithm with base $e \approx 2.718$, and $\exp\{x\} = e^x$ is the exponential function. Some sources write $\ln x$ for the natural logarithm and reserve $\log x$ for base 10.

Theorem 0.15 (exp{} and log{} are mutual inverses)**.**

1. *For every* $a > 0$*:* $\exp\{\log\{a\}\} = a$*.*
2. *For every* $a \in \mathbb{R}$*:* $\log\{\exp\{a\}\} = a$*.*

Theorem 0.16 (Logarithm of a product)**.** *If* $a > 0$ *and* $b > 0$*, then*

$$\log\{a \cdot b\} = \log\{a\} + \log\{b\}$$

Corollary 0.1 (Logarithm of a quotient)**.** *If* $a > 0$ *and* $b > 0$*, then*

$$\log\left\{\frac{a}{b}\right\} = \log\{a\} - \log\{b\}$$

Proof

Proof. Since $a > 0$ and $b > 0$, the quotient $\frac{a}{b}$ is positive, so Theorem 0.16 applies to the product $\frac{a}{b} \cdot b$:

$$\begin{aligned} \log\{a\} &= \log\left\{\frac{a}{b} \cdot b\right\} && (a = \tfrac{a}{b} \cdot b) \\ &= \log\left\{\frac{a}{b}\right\} + \log\{b\} && (\text{logarithm of a product}) \end{aligned}$$

The second step applies Theorem 0.16. Subtracting $\log\{b\}$ from both sides gives $\log\{a\} - \log\{b\} = \log\{\frac{a}{b}\}$. □

Theorem 0.17 (Logarithm of a power)**.** *If* $a > 0$ *and* $b \in \mathbb{R}$*, then*

$$\log\{a^b\} = b \cdot \log\{a\}$$

Theorem 0.18 (exponential of a sum)**.**
The exponential of a sum is equal to the product of the exponentials of the addends:

$$\exp\{a + b\} = \exp\{a\} \cdot \exp\{b\}$$

Corollary 0.2 (exponential of a difference)**.**
The exponential of a difference is the exponential of the first term divided by the exponential of the second term:

$$\exp\{a-b\} = \frac{\exp\{a\}}{\exp\{b\}}$$

Theorem 0.19 (Power of a power)**.** *If $a > 0$ and $b, c \in \mathbb{R}$, then*

$$a^{bc} = (a^b)^c = (a^c)^b$$

Example 0.4 (A negative base)**.** With $a = -1$, $b = 2$, and $c = \frac{1}{2}$:

$$\begin{aligned} a^{bc} &= (-1)^{2\cdot\frac{1}{2}} \\ &= (-1)^1 \\ &= -1 \end{aligned}$$

but

$$\begin{aligned} (a^b)^c &= ((-1)^2)^{\frac{1}{2}} \\ &= 1^{\frac{1}{2}} \\ &= 1 \end{aligned}$$

So $a^{bc} \neq (a^b)^c$ here, which is why Theorem 0.19 requires $a > 0$. The third expression, $(a^c)^b = \left((-1)^{\frac{1}{2}}\right)^2$, is not even a real number.

Corollary 0.3 (natural exponential of a product)**.**

$$\exp\{ab\} = (\exp\{a\})^b = (\exp\{b\})^a$$

Exercise 0.1. For $b, c \in \mathbb{R}$, when does $b^c = bc$?

Solution

Solution 0.1. We only count a pair (b, c) when b^c is a real number, so for $b < 0$ we only consider integer c (R agrees: `(-8)^(1/3)` is `NaN`). With that convention, $bc = b^c$ in each of the following cases:

1. $c = 1$, for every b.
2. $b = 0$ and $c > 0$, since then $b^c = 0 = bc$. ($b = 0$ and $c = 0$ fails, since $0^0 = 1 \neq 0$.)

3. $b > 0$, $c > 0$, $c \neq 1$, and $b = \exp\left\{\frac{\log\{c\}}{c-1}\right\}$. For $b > 0$, dividing both sides of $b^c = bc$ by b gives $b^{c-1} = c$, which needs $c > 0$ because $b^{c-1} > 0$; taking logarithms then gives $(c-1)\log\{b\} = \log\{c\}$. For example, $c = 2$ gives $b = 2$, and indeed $2^2 = 4 = 2 \cdot 2$.
4. $b < 0$ and c is an odd integer with $c \geq 3$, with $b = -c^{1/(c-1)}$; for example, $b = -\sqrt{3}$ and $c = 3$ give $b^c = -3\sqrt{3} = bc$.
5. $b < 0$ and c is an even integer with $c \leq -2$, with $b = -(-c)^{1/(c-1)}$; for example, $b = -2^{-1/3}$ and $c = -2$ give $b^c = 2^{2/3} = bc$.

For $b < 0$, cases 4 and 5 come from $b^{c-1} = c$ as well: when $c-1$ is even, $b^{c-1} > 0$, so c must be positive; when $c-1$ is odd, $b^{c-1} < 0$, so c must be negative.
See the red contours in Figure 2 for a visualization of the $b \geq 0$ cases.

(This 3D surface is interactive; see the HTML or slide version of this page to view it.)

Figure 1: Graph of $b * c$ and b^c

(This 3D surface is interactive; see the HTML or slide version of this page to view it.)

Figure 2: **Graph of** $b^c - b * c$. Red contour lines show where $b^c = b * c$.

Exercise 0.2. For $a \geq 0,\ b, c \in \mathbb{R}$, when does $(a^b)^c = a^{(b^c)}$?

Solution

Solution 0.2. Short answer: rarely (that's all you need to know for this course).
Long answer:
Split on whether $a > 0$ or $a = 0$, because the logarithm we use for $a > 0$ is undefined at $a = 0$.
Case $a > 0$**.** By Theorem 0.19, $(a^b)^c = a^{bc}$, so the question becomes when $a^{bc} = a^{(b^c)}$ (for pairs (b, c) where b^c is defined). Because $a > 0$, both sides are positive, and we can take logarithms (Theorem 0.17):

$$\begin{aligned} a^{bc} &= a^{(b^c)} \\ \log\{a^{bc}\} &= \log\left\{a^{(b^c)}\right\} \quad \text{(take logarithms of both sides)} \\ bc \cdot \log\{a\} &= b^c \cdot \log\{a\} \quad \text{(logarithm of a power)} \end{aligned} \tag{1}$$

The last line of Equation 1 holds exactly when

1. $a = 1$ (so that $\log\{a\} = 0$), or
2. $bc = b^c$ (see Exercise 0.1).

Case $a = 0$. Here we cannot take logarithms, so we work from the values of powers of 0: $0^s = 0$ for $s > 0$, $0^0 = 1$, and 0^s is undefined for $s < 0$.

- If $b < 0$, then 0^b is undefined, so $(a^b)^c$ is undefined.
- If $b > 0$, then $(0^b)^c = 0^c$ and $b^c > 0$, so $0^{(b^c)} = 0$; the two sides agree exactly when $c > 0$.
- If $b = 0$, then $(0^0)^c = 1^c = 1$; for $c > 0$, $0^{(0^c)} = 0^0 = 1$, so the two sides agree, and for $c \leq 0$ they do not.

So for $a = 0$, $(a^b)^c = a^{(b^c)}$ exactly when $b \geq 0$ and $c > 0$.
In particular, when $a = 0$, $b \geq 0$, and $c = 0$, the two sides differ:

$$\begin{aligned}(a^b)^c &= (0^b)^0 \\ &= 1 \\ a^{(b^c)} &= 0^{(b^0)} \\ &= 0^1 \\ &= 0\end{aligned}$$

Rudin, Walter. 1976. *Principles of Mathematical Analysis*. 3rd ed. International Series in Pure and Applied Mathematics. McGraw-Hill.