

Polynomials, Exponentials and Logarithms

Last modified: 2026-10-27 20:53:54 (PDT)

Polynomials

Exercise 0.1 (Reading a polynomial). Let $f(x) = 3x^4 - x + 7$.

1. Write f in the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, listing every coefficient a_4, a_3, a_2, a_1, a_0 .
2. What is the degree of f , and what is its leading coefficient?

Solution

Solution 0.1.

1. $f(x) = 3x^4 + 0 \cdot x^3 + 0 \cdot x^2 + (-1) \cdot x + 7$, so $a_4 = 3$, $a_3 = 0$, $a_2 = 0$, $a_1 = -1$, and $a_0 = 7$. Powers of x that do not appear have coefficient 0.
2. The highest power with a nonzero coefficient is x^4 , so the degree is 4 and the leading coefficient is $a_4 = 3$.

Definition 0.1 (Constant function). A **function** $f : A \rightarrow B$ is **constant** if there is some $c \in B$ with $f(x) = c$ for all $x \in A$.

Example 0.1 (Constant and non-constant functions).

- $f(x) = 7$ on $\mathbb{R}$ is constant, with $c = 7$.
- $f(x) = x^2$ on $\mathbb{R}$ is not constant: $f(0) = 0$ but $f(1) = 1$.

Definition 0.2 (Polynomial). A **polynomial** (in one variable x) [1] is a **function** $f : \mathbb{R} \rightarrow \mathbb{R}$ that can be written as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some integer $n \geq 0$ and constants $a_0, a_1, \dots, a_n \in \mathbb{R}$ with $a_n \neq 0$. The **terms** of f are $a_n x^n, \dots, a_1 x, a_0$.

i Remark (Constant and zero polynomials)

Remark 0.1 (Constant and zero polynomials). A **constant function** $f(x) = 7$ is a polynomial with $n = 0$ and $a_0 = 7$. The requirement $a_n \neq 0$ means this definition covers nonzero polynomials only: the zero function $f(x) = 0$ is excluded here, because it has no nonzero coefficient to serve as a_n .

Definition 0.3 (Coefficients of a polynomial). The constants $a_0, a_1, \dots, a_n$ of a polynomial f (Definition 0.2) are the **coefficients** of f , with a_k the coefficient of x^k .

Definition 0.4 (Degree of a polynomial). Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a **polynomial** with $a_n \neq 0$. The integer n is the **degree** of f .

Example 0.2 (Degrees of some polynomials).

- $f(x) = 4x$ has degree 1.
- $f(x) = 5 - 2x^2 + x^3$ has degree 3: the degree is the highest power with a nonzero coefficient, not the power in the first term written.
- A **constant** polynomial $f(x) = 7$ has degree 0.

Definition 0.5 (Leading coefficient). Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a **polynomial** of **degree** n . The constant a_n is the **leading coefficient** of f .

Example 0.3 (Leading coefficients).

- $f(x) = 5 - 2x^2 + x^3$ has degree 3, so its leading coefficient is $a_3 = 1$, not the 5 written first.
- $f(x) = 3 - x^2$ has leading coefficient $a_2 = -1$.

Definition 0.6 (Quadratic polynomial). A **polynomial** of **degree** 2, $f(x) = a_2x^2 + a_1x + a_0$ with $a_2 \neq 0$, is a **quadratic** [2] polynomial.

Definition 0.7 (Cubic polynomial). A **polynomial** of **degree** 3 is a **cubic** polynomial.

Example 0.4 (Quadratic and cubic polynomials).

•

$$\begin{aligned} f(x) &= (x - 2)^2 \\ &= x^2 - 4x + 4 \end{aligned}$$

is quadratic, by **Remark 5 in Convexity, Infimum and Sums** with $a = x$ and $b = 2$.

- $f(x) = x^3 - 3x$ is cubic.
- $f(x) = 4x + 1$ is neither: it has degree 1.

Definition 0.8 (Parabola). A **parabola** [3] is the **graph** of a quadratic polynomial (Definition 0.6).

Example 0.5 (The parabola $y = x^2$). The graph of $f(x) = x^2$ is a parabola. It contains the points $(-1, 1)$, $(0, 0)$ and $(2, 4)$, and its lowest point is $(0, 0)$, since $x^2 \geq 0$ for every x .

Definition 0.9 (Monomial (power product)). A **monomial**, also called a **power product** [4], in the variables $x_1, \dots, x_p$ is an **expression**

$$x_1^{a_1} x_2^{a_2} \dots x_p^{a_p} \tag{1}$$

that multiplies **powers** of those variables, whose **exponents** $a_1, \dots, a_p$ are nonnegative **integers**.

Each $a_j = 0$ contributes the factor $x_j^0 = 1$ (**zero exponent**).

Example 0.6 (Which expressions are monomials). Take the four variables x_1, x_2, x_3, x_4 .

- $x_1^3 x_4$ is a **monomial**, with **exponents** $(a_1, a_2, a_3, a_4) = (3, 0, 0, 1)$.
- The **constant** 1 is the monomial whose exponents are all 0.
- $x_1/x_2 = x_1 x_2^{-1}$ is not a monomial, because the **exponent** of x_2 is negative.
- $\sqrt{x_3} = x_3^{1/2}$ is not a monomial, because its exponent is not an **integer**.
- $x_1 + x_4$ is not a monomial: it adds two monomials rather than multiplying **powers**.

Definition 0.10 (Degree of a monomial). The **degree** of the **monomial** $x_1^{a_1}x_2^{a_2}\cdots x_p^{a_p}$ is the **sum** of its **exponents**, $a_1 + a_2 + \cdots + a_p$ [4].

Example 0.7 (Degrees of some monomials).

- $x_1^3x_4$ has **degree** $3 + 1 = 4$.
- $x_2^2x_3^5$ has **degree** $2 + 5 = 7$.
- The **constant monomial** 1 has **degree** 0.
- With one variable, the monomial x^n has **degree** n , which is the **degree** of the one-term **polynomial** $f(x) = x^n$.

Affine and linear functions of one variable

Definition 0.11 (Affine function of one variable). A **function** $f : \mathbb{R} \rightarrow \mathbb{R}$ is **affine** [5] if there are numbers m and b such that

$$f(x) = mx + b \quad \text{for all } x \in \mathbb{R}$$

Definition 0.12 (Slope). In the formula $f(x) = mx + b$ for an affine function f (Definition 0.11), m is called the **slope** of f [6].

Definition 0.13 (Intercept). In the formula $f(x) = mx + b$ for an affine function f (Definition 0.11), b is called the **intercept** of f .

Definition 0.14 (Linear function of one variable). A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is **linear** [7] if it is affine (Definition 0.11) with intercept 0; that is, if there is a number m such that

$$f(x) = mx \quad \text{for all } x \in \mathbb{R}$$

Theorem 0.1 (Intercept and slope of an affine function). *Let $f(x) = mx + b$ be an affine function (Definition 0.11).*

1. *The intercept is the value at zero: $f(0) = b$.*
2. *The slope is the change in f per unit change in x : for any $x_1 \neq x_2$,*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = m$$

3. So f determines its slope and intercept: if also $f(x) = m'x + b'$ for all x , then $m' = m$ and $b' = b$.

i Proof

Proof. **Part 1.**

$$\begin{aligned}
 f(0) &= m \cdot 0 + b && \text{(Definition 0.11)} \\
 &= 0 + b && \text{(any number times 0 is 0)} \\
 &= b + 0 && \text{(Theorem 5 in Convexity, Infimum and Sums)} \\
 &= b && \text{(Theorem 4 in Convexity, Infimum and Sums)}
 \end{aligned}$$

Part 2. First the numerator:

$$\begin{aligned}
 f(x_2) - f(x_1) &= (mx_2 + b) - (mx_1 + b) && \text{(Definition 0.11)} \\
 &= mx_2 + b - mx_1 - b && \text{(subtracting a sum subtracts each term)} \\
 &= mx_2 - mx_1 + b - b && \text{(Theorem 5 in Convexity, Infimum and Sums, swapping } b \text{ and } -b) \\
 &= (mx_2 - mx_1) + (b - b) && \text{(Theorem 6 in Convexity, Infimum and Sums)} \\
 &= (mx_2 - mx_1) + 0 && (b - b = 0) \\
 &= mx_2 - mx_1 && \text{(Theorem 4 in Convexity, Infimum and Sums)} \\
 &= mx_2 + m(-x_1) && (-mx_1 = m(-x_1), \text{ by Theorem 6 in Equalities, Inequalities and Sums}) \\
 &= m(x_2 + (-x_1)) && \text{(Theorem 11 in Convexity, Infimum and Sums, read right to left)} \\
 &= m(x_2 - x_1) && \text{(adding a negative is subtracting)}
 \end{aligned}$$

Then, writing $d = x_2 - x_1$, which is not 0 because $x_1 \neq x_2$,

$$\begin{aligned}
 \frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{md}{d} && \text{(the numerator above)} \\
 &= (md) \cdot \frac{1}{d} && \text{(Theorem 10 in Convexity, Infimum and Sums)} \\
 &= m \cdot \left(d \cdot \frac{1}{d}\right) && \text{(Theorem 9 in Convexity, Infimum and Sums)} \\
 &= m \cdot \frac{d}{d} && \text{(Theorem 10 in Convexity, Infimum and Sums)} \\
 &= m \cdot 1 && \text{(a nonzero number divided by itself is 1)} \\
 &= m && \text{(Theorem 7 in Convexity, Infimum and Sums)}
 \end{aligned}$$

Part 3. Part 1, applied to each formula, gives

$$\begin{aligned}
 b &= f(0) \\
 &= b'.
 \end{aligned}$$

Part 2 with $x_1 = 0$ and $x_2 = 1$, applied to each formula, gives

$$\begin{aligned} m &= f(1) - f(0) \\ &= m'. \end{aligned}$$

□

Example 0.8 (An affine function, and a linear one). $f(x) = 2x + 3$ is affine with slope 2 and intercept 3. By Theorem 0.1, $f(0) = 3$, and each unit step in x raises f by 2: for example, $f(1) = 5$ and $f(2) = 7$. Its intercept is not 0, so it is not linear.

$g(x) = -\frac{1}{2}x$ is affine with intercept 0, so it is linear: $g(0) = 0$, and each unit step in x lowers g by $\frac{1}{2}$.

i Remark (Elementary algebra calls $mx + b$ “linear”)

Remark 0.2 (Elementary algebra calls $mx + b$ “linear”). Elementary algebra usually calls $f(x) = mx + b$ a *linear function*, because its graph is a straight line. This site calls that an affine function and keeps “linear” for the case $b = 0$, because that is the sense used in linear algebra, where a [linear map](#) must send 0 to 0 ([theorem](#)).

Limits of sequences

Definition 0.15 (Limit of a sequence). A real number L is the **limit** of a [sequence](#) (a_n) of real numbers [8], written $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow L$, if for every $\varepsilon > 0$ there is a natural number N such that

$$|a_n - L| < \varepsilon \quad \text{for every } n \geq N.$$

Example 0.9 (The limit of a sequence). $a_n = \frac{1}{n}$ has limit 0. Given $\varepsilon > 0$, take N to be any natural number larger than $\frac{1}{\varepsilon}$. For every $n \geq N$, $|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. For example, with $\varepsilon = 0.01$, take $N = 101$: every $n \geq 101$ has $\frac{1}{n} \leq \frac{1}{101} < 0.01$.

Definition 0.16 (Convergent sequence (converges)). A sequence (a_n) of real numbers **converges** to a real number L if L is its limit (Definition 0.15). A sequence that converges to some real number is **convergent**.

Example 0.10 (A convergent sequence). $a_n = \frac{1}{n}$ converges to 0, because 0 is its limit (Example 0.9), so $a_n = \frac{1}{n}$ is convergent.

Definition 0.17 (Divergent sequence (diverges)). A sequence that does not converge (Definition 0.16) to any real number **diverges**, and a sequence that diverges is **divergent**.

Example 0.11 (A divergent sequence). $c_n = n$ diverges. For any real number L and $\varepsilon = 1$, every $n > L + 1$ has $|n - L| > 1$, so no N works. For example, with $L = 5$, every $n \geq 7$ has $|n - 5| \geq 2$. It diverges to ∞ : for every real number M , every $n \geq N$ has $c_n = n > M$ when N is a natural number larger than M .

Definition 0.18 (Divergence of a sequence to infinity). A [sequence](#) (a_n) of real numbers **diverges to** ∞ , written $\lim_{n \rightarrow \infty} a_n = \infty$, if for every real number M there is a natural number N such that $a_n > M$ for every $n \geq N$.

Series

Definition 0.19 (Non-decreasing sequence). A [sequence](#) $a_1, a_2, \dots$ of real numbers or of [extended non-negative reals](#) is **non-decreasing** if each term is at least as large as the one before it:

$$a_n \leq a_{n+1} \quad \text{for every } n \in \mathbb{N}$$

Example 0.12 (Non-decreasing and not).

- $1, 2, 2, 3, 3, 3, \dots$ is non-decreasing: a term may equal the one before it.
- $1, 2, \infty, \infty, \dots$ is non-decreasing, because ∞ is greater than every real number.
- $1, \frac{1}{2}, \frac{1}{3}, \dots$ is not non-decreasing, because

$$\begin{aligned} a_2 &= \frac{1}{2} \\ &< 1 \\ &= a_1. \end{aligned}$$

Definition 0.20 (Partial sum). Let $a_1, a_2, \dots$ be a [sequence](#) of real numbers or of [extended non-negative reals](#). For each $n \in \mathbb{N}$, its n th **partial sum** [9] is the [sum](#) of its first n terms:

$$s_n \stackrel{\text{def}}{=} \sum_{i=1}^n a_i = a_1 + \dots + a_n$$

Example 0.13 (Partial sums of halves). For $a_i = 1/2^i$, the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$, the first three partial sums are $s_1 = \frac{1}{2}$,

$$\begin{aligned} s_2 &= \frac{1}{2} + \frac{1}{4} \\ &= \frac{3}{4}, \end{aligned}$$

and

$$\begin{aligned} s_3 &= \frac{3}{4} + \frac{1}{8} \\ &= \frac{7}{8}. \end{aligned}$$

In general, $s_n = 1 - 1/2^n$.

Definition 0.21 (Infinite sum, series). Let $a_1, a_2, \dots$ be a sequence in $[0, \infty]$, with partial sums s_n (Definition 0.20). The **infinite sum** (also called a **series**) [9] $\sum_{i=1}^{\infty} a_i$ is the limit of the partial sums:

$$\sum_{i=1}^{\infty} a_i \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} s_n$$

Here the limit is a real number when the partial sums are real numbers that [converge](#) to it, and it is ∞ when the partial sums [diverge to \$\infty\$](#) or some partial sum is ∞ . Sums with other index ranges, such as $\sum_{k=0}^{\infty} a_k$, are defined the same way.

Example 0.14 (A finite and an infinite sum).

- For $a_i = 1/2^i$, the partial sums $s_n = 1 - 1/2^n$ (Example 0.13) converge to 1, because $1/2^n \rightarrow 0$. So $\sum_{i=1}^{\infty} 1/2^i = 1$.
- For $a_i = 1$, the partial sums are $s_n = n$, which diverge to ∞ . So $\sum_{i=1}^{\infty} 1 = \infty$.

Lemma 0.1 (Sums of non-negative terms). *Let $a_1, a_2, \dots$ be values in $[0, \infty]$, with partial sums s_n (Definition 0.20), where $x + \infty = \infty$ for every x in $[0, \infty]$. Then the partial sums are **non-decreasing**, $s_1 \leq s_2 \leq \dots$, so their **limit**, the infinite sum $\sum_{i=1}^{\infty} a_i$ (Definition 0.21), always exists, and it is either a finite number or ∞ .*

i Proof

Proof. For each n :

$$\begin{aligned} s_{n+1} &= s_n + a_{n+1} && \text{(definition of } s_{n+1}) \\ &\geq s_n && (a_{n+1} \geq 0) \end{aligned}$$

If some partial sum s_N is ∞ , then $s_n = \infty$ for every $n \geq N$, so the limit is ∞ . Otherwise, the partial sums form a non-decreasing sequence of **real numbers**. If that sequence is **bounded above**, it **converges** to a finite number [10]. If it is not bounded above, then for every number M some s_N exceeds M , and so does every later $s_n \geq s_N$; the limit is ∞ . $\square$

Exponentials and Logarithms

Definition 0.22 (Exponential function). The **exponential function** [11] $\exp : \mathbb{R} \rightarrow (0, \infty)$ is

$$\exp\{x\} \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n,$$

the **limit** of the **sequence** $(1 + \frac{x}{1})^1, (1 + \frac{x}{2})^2, (1 + \frac{x}{3})^3, \dots$. That limit exists for every real number x , and it is positive.

Example 0.15 (Approximating $\exp\{1\}$). For $x = 1$, the terms $(1 + \frac{1}{n})^n$ are:

n	$(1 + \frac{1}{n})^n$
1	2
10	2.59374 ...
100	2.70481 ...
1000	2.71692 ...

They approach $\exp\{1\} = 2.71828 \dots$. For $x = 0$, every term is $(1 + 0)^n = 1$, so $\exp\{0\} = 1$.

Definition 0.23 (Euler’s number). **Euler’s number** [12] is

$$e \stackrel{\text{def}}{=} \exp\{1\} = 2.71828 \dots$$

Example 0.16 (Euler’s number is irrational). $e = 2.71828 \dots$ is an **irrational number**, so no fraction equals it exactly. The fraction $\frac{19}{7} = 2.714 \dots$ is close, but $\frac{19}{7} \neq e$ [12].

Definition 0.24 (Natural logarithm). For a real number $a > 0$, the **natural logarithm** of a , written $\log\{a\}$ [13], is the real number y with $\exp\{y\} = a$. The exponential function takes each positive value at exactly one input, so there is exactly one such y , and $\log : (0, \infty) \rightarrow \mathbb{R}$ is the **inverse function** of $\exp$ (Definition 0.22).

i Remark (Other notations for the natural logarithm)

Remark 0.3 (Other notations for the natural logarithm). In these notes, $\log\{x\}$ is always the natural logarithm, the logarithm with base e (Definition 0.23) [14]. Some sources write $\ln x$ for the natural logarithm and reserve $\log x$ for the logarithm with base 10.

Example 0.17 (Natural logarithms).

- $\log\{1\} = 0$, because $\exp\{0\} = 1$ (Example 0.15).
- $\log\{e\} = 1$, because $\exp\{1\} = e$ (Definition 0.23).
- $\log\{0\}$ and $\log\{-2\}$ are not defined, because $\exp\{y\} > 0$ for every real y .

Definition 0.25 (Power with a real exponent). For a real number $a > 0$ and a real number b ,

$$a^b \stackrel{\text{def}}{=} \exp\{b \cdot \log\{a\}\}.$$

When b is an integer, this agrees with [Definition 5 in Equalities, Inequalities and Minimizers](#) and [Definition 9 in Equalities, Inequalities and Minimizers](#).

Example 0.18 (Real powers, and e^x).

- $2^3 = \exp\{3 \cdot \log\{2\}\} \approx \exp\{3 \cdot 0.69315\} \approx \exp\{2.07944\} \approx 8$, which agrees with

$$\begin{aligned} 2^3 &= 2 \cdot 2 \cdot 2 \\ &= 8 \end{aligned}$$

from [Definition 5 in Equalities, Inequalities and Minimizers](#).

- $2^{1/2} = \exp\{\frac{1}{2} \cdot \log\{2\}\} \approx \exp\{0.34657\} \approx 1.41421$, which is $\sqrt{2}$ ([Definition 10 in Equalities, Inequalities and Minimizers](#)).
- For every real x , with base e ([Definition 0.23](#)):

$$\begin{aligned} e^x &= \exp\{x \cdot \log\{e\}\} && \text{(Definition 0.25, with } a = e \text{ and } b = x) \\ &= \exp\{x \cdot 1\} && (\log\{e\} = 1, \text{ by Example 0.17}) \\ &= \exp\{x\} && \text{(Theorem 7 in Convexity, Infimum and Sums)} \end{aligned}$$

So e^x and $\exp\{x\}$ are two names for the same number.

Definition 0.26 (Real powers of zero and of negative numbers). [Definition 5 in Equalities, Inequalities and Minimizers](#), [Definition 9 in Equalities, Inequalities and Minimizers](#) and [Definition 0.25](#) leave some powers b^c with $b \leq 0$ unassigned. We complete them as follows.

- For a real number $c > 0$, $0^c \stackrel{\text{def}}{=} 0$. For a natural number c , this agrees with [Definition 5 in Equalities, Inequalities and Minimizers](#).
- For $b < 0$, b^c is defined only when c is an integer, by [Definition 5 in Equalities, Inequalities and Minimizers](#) and [Definition 9 in Equalities, Inequalities and Minimizers](#); for $b < 0$ and a non-integer c , b^c is undefined.

Example 0.19 (Powers of zero and of negative numbers).

- $0^{1/2} = 0$, which agrees with $\sqrt{0} = 0$ ([Definition 10 in Equalities, Inequalities and Minimizers](#)).
- $(-8)^2 = 64$ and $(-8)^{-1} = -\frac{1}{8}$ are defined, since the exponents are integers.
- $(-8)^{1/3}$ is undefined, even though $(-2)^3 = -8$. R follows the same convention: $(-8) \wedge (1/3)$ is NaN.

Theorem 0.2 (exp and log are mutual inverses).

1. For every $a > 0$: $\exp\{\log\{a\}\} = a$.
2. For every $a \in \mathbb{R}$: $\log\{\exp\{a\}\} = a$.

Theorem 0.3 (Logarithm of a product). If $a > 0$ and $b > 0$, then

$$\log\{a \cdot b\} = \log\{a\} + \log\{b\}$$

Corollary 0.1 (Logarithm of a quotient). *If $a > 0$ and $b > 0$, then*

$$\log\left\{\frac{a}{b}\right\} = \log\{a\} - \log\{b\}$$

i Proof

Proof. Since $a > 0$ and $b > 0$, the quotient $\frac{a}{b}$ is positive, so Theorem 0.3 applies to the product $\frac{a}{b} \cdot b$:

$$\begin{aligned}\log\{a\} &= \log\left\{\frac{a}{b} \cdot b\right\} && (a = \frac{a}{b} \cdot b) \\ &= \log\left\{\frac{a}{b}\right\} + \log\{b\} && (\text{logarithm of a product})\end{aligned}$$

The second step applies Theorem 0.3. Subtracting $\log\{b\}$ from both sides gives $\log\{a\} - \log\{b\} = \log\left\{\frac{a}{b}\right\}$. $\square$

Theorem 0.4 (Logarithm of a power). *If $a > 0$ and $b \in \mathbb{R}$, then*

$$\log\{a^b\} = b \cdot \log\{a\}$$

Theorem 0.5 (Exponential of a sum). *The exponential of a sum is equal to the product of the exponentials of its terms:*

$$\exp\{a + b\} = \exp\{a\} \cdot \exp\{b\}$$

Example 0.20 (Exponential of a sum). With $a = 2$ and $b = 3$, $\exp\{2 + 3\} = \exp\{5\} \approx 148.41$, and $\exp\{2\} \cdot \exp\{3\} \approx 7.3891 \cdot 20.0855 \approx 148.41$.

Corollary 0.2 (Exponential of a difference). *The exponential of a difference is the exponential of the first term divided by the exponential of the second term:*

$$\exp\{a - b\} = \frac{\exp\{a\}}{\exp\{b\}}$$

Example 0.21 (Exponential of a difference). With $a = 5$ and $b = 2$, $\exp\{5 - 2\} = \exp\{3\} \approx 20.09$, and $\frac{\exp\{5\}}{\exp\{2\}} \approx \frac{148.4132}{7.3891} \approx 20.09$.

Theorem 0.6 (Powers of 1 and first powers). *For every $b \in \mathbb{R}$,*

$$1^b = 1,$$

and for every $a \in \mathbb{R}$,

$$a^1 = a.$$

Theorem 0.7 (Power of a sum). *If $a > 0$ and $b, c \in \mathbb{R}$, then*

$$a^{b+c} = a^b \cdot a^c$$

Example 0.22 (Power of a sum). With $a = 2$, $b = 3$, and $c = 4$,

$$\begin{aligned} 2^{3+4} &= 2^7 \\ &= 128, \end{aligned}$$

and

$$\begin{aligned} 2^3 \cdot 2^4 &= 8 \cdot 16 \\ &= 128. \end{aligned}$$

Theorem 0.8 (Power of a product). *If $a, b > 0$ and $c \in \mathbb{R}$, then*

$$(ab)^c = a^c \cdot b^c$$

When c is a positive integer, the same identity holds for all $a, b \in \mathbb{R}$, because both sides are products of c copies of a and c copies of b , which can be regrouped by [Theorem 8 in Convexity, Infimum and Sums](#) and [Theorem 9 in Convexity, Infimum and Sums](#).

Example 0.23 (Power of a product). With $a = 2$, $b = 3$, and $c = 2$,

$$\begin{aligned} (2 \cdot 3)^2 &= 6^2 \\ &= 36, \end{aligned}$$

and

$$\begin{aligned} 2^2 \cdot 3^2 &= 4 \cdot 9 \\ &= 36. \end{aligned}$$

Theorem 0.9 (Power of a power). *If $a > 0$ and $b, c \in \mathbb{R}$, then*

$$\begin{aligned} a^{bc} &= (a^b)^c \\ &= (a^c)^b \end{aligned}$$

Example 0.24 (A negative base). With $a = -1$, $b = 2$, and $c = \frac{1}{2}$:

$$\begin{aligned} a^{bc} &= (-1)^{2 \cdot \frac{1}{2}} \\ &= (-1)^1 \\ &= -1 \end{aligned}$$

but

$$\begin{aligned} (a^b)^c &= ((-1)^2)^{\frac{1}{2}} \\ &= 1^{\frac{1}{2}} \\ &= 1 \end{aligned}$$

So $a^{bc} \neq (a^b)^c$ here, which is why Theorem 0.9 requires $a > 0$. The third expression, $(a^c)^b = ((-1)^{\frac{1}{2}})^2$, is not even a real number.

Corollary 0.3 (Natural exponential of a product).

$$\begin{aligned} \exp\{ab\} &= (\exp\{a\})^b \\ &= (\exp\{b\})^a \end{aligned}$$

i Remark (Tarski’s high school identities)

Remark 0.4 (Tarski’s high school identities). Restricted to positive integers, the following results are Tarski’s eleven “high school” identities [15]:

- sums are symmetric and associative ([Theorem 5 in Convexity, Infimum and Sums](#), [Theorem 6 in Convexity, Infimum and Sums](#));
- multiplying by 1 changes nothing, products are symmetric and associative, and multiplication is distributive ([Theorem 7 in Convexity, Infimum and Sums](#), [Theorem 8 in Convexity, Infimum and Sums](#), [Theorem 9 in Convexity, Infimum and Sums](#), [Theorem 11 in Convexity, Infimum and Sums](#));
- $1^b = 1$, $a^1 = a$, and the power of a sum, of a product, and of a power ([Theorem 0.6](#), [Theorem 0.7](#), [Theorem 0.8](#), [Theorem 0.9](#)).

i Remark (Tarski's identities are not complete)

Remark 0.5 (Tarski's identities are not complete). Tarski asked whether every identity in $+$, $\times$, exponentiation and 1 that is true for all positive integers can be derived from the eleven identities in Remark 0.4. It cannot: Wilkie found an identity that is true for all positive integers but does not follow from them. So this list is a useful core, not a complete rulebook.

Definition 0.27 (Contour line). For a function g of two real variables b and c and a number k , the **contour line** [16] of g at height k is the set of points where g equals k :

$$\{(b, c) : g(b, c) = k\}$$

Example 0.25 (Contour lines of a bowl). For $g(b, c) = b^2 + c^2$, the contour line at height $k = 4$ is $\{(b, c) : b^2 + c^2 = 4\}$, the circle of radius 2 around $(0, 0)$: for example,

$$\begin{aligned} g(2, 0) &= 4 + 0 \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} g(0, -2) &= 0 + 4 \\ &= 4. \end{aligned}$$

The contour line at height $k = -1$ is empty, because $b^2 + c^2 \geq 0$ for all b and c .



Figure 1: The contour lines of $g(b, c) = b^2 + c^2$ at heights 1, 4 and 9. The dots mark the points $(2, 0)$ and $(0, -2)$ of the example on the circle at height 4.

i Remark (Contour lines on a plot of a surface)

Remark 0.6 (Contour lines on a plot of a surface). On a plot of the surface $z = g(b, c)$, the **contour line** is the curve along which the surface has height k .

Exercise 0.2 (Exponentiation versus multiplication). For $b, c \in \mathbb{R}$, when does $b^c = bc$?

Solution

Solution 0.2. We only count a pair (b, c) when b^c is defined, so for $b < 0$ we only consider integer c (Definition 0.26). With that convention, $bc = b^c$ in each of the following cases:

1. $c = 1$, for every b .

2. $b = 0$ and $c > 0$, since then

$$\begin{aligned} b^c &= 0 \\ &= bc. \end{aligned}$$

($b = 0$ and $c = 0$ fails, since $0^0 = 1 \neq 0$.)

3. $b > 0$, $c > 0$, $c \neq 1$, and $b = \exp\left\{\frac{\log\{c\}}{c-1}\right\}$. For $b > 0$, dividing both sides of $b^c = bc$ by b gives $b^{c-1} = c$, which needs $c > 0$ because $b^{c-1} > 0$; taking logarithms then gives $(c-1)\log\{b\} = \log\{c\}$. For example, $c = 2$ gives $b = 2$, and indeed

$$\begin{aligned} 2^2 &= 4 \\ &= 2 \cdot 2. \end{aligned}$$

4. $b < 0$ and c is an odd integer with $c \geq 3$, with $b = -c^{1/(c-1)}$; for example, $b = -\sqrt{3}$ and $c = 3$ give

$$\begin{aligned} b^c &= -3\sqrt{3} \\ &= bc. \end{aligned}$$

5. $b < 0$ and c is an even integer with $c \leq -2$, with $b = -(-c)^{1/(c-1)}$; for example, $b = -2^{-1/3}$ and $c = -2$ give

$$\begin{aligned} b^c &= 2^{2/3} \\ &= bc. \end{aligned}$$

For $b < 0$, cases 4 and 5 come from $b^{c-1} = c$ as well: when $c-1$ is even, $b^{c-1} > 0$, so c must be positive; when $c-1$ is odd, $b^{c-1} < 0$, so c must be negative. See the red contours in Figure 3 for a visualization of the $b \geq 0$ cases.

(This 3D surface is interactive; see the HTML or slide version of this page to view it.)

Figure 2: Graph of $b * c$ and b^c

(This 3D surface is interactive; see the HTML or slide version of this page to view it.)

Figure 3: **Graph of $b^c - b * c$.** The red **contour lines** are at heights 0 and 1; the ones at height 0 show where $b^c = b * c$.

Theorem 0.10 (When exponentiation equals multiplication). *Let $b, c \in \mathbb{R}$, where b^c is defined (for $b < 0$, only integer c is allowed; Definition 0.26). Then $b^c = bc$ if and only if*

one of the following holds:

1. $c = 1$;
2. $b = 0$ and $c > 0$;
3. $b > 0$, $c > 0$, $c \neq 1$, and $b = \exp\left\{\frac{\log\{c\}}{c-1}\right\}$;
4. $b < 0$, c is an odd integer with $c \geq 3$, and $b = -c^{1/(c-1)}$;
5. $b < 0$, c is an even integer with $c \leq -2$, and $b = -(-c)^{1/(c-1)}$.

Proof

Proof. This is the solution to Exercise 0.2, worked in Solution 0.2. □

Exercise 0.3 (Repeated exponentiation). For $a \geq 0$, $b, c \in \mathbb{R}$, when does $(a^b)^c = a^{(b^c)}$?

Solution

Solution 0.3. Short answer: rarely (that's all you need to know for this course).

Long answer:

Split on whether $a > 0$ or $a = 0$, because the logarithm we use for $a > 0$ is undefined at $a = 0$.

Case $a > 0$. By Theorem 0.9, $(a^b)^c = a^{bc}$, so the question becomes when $a^{bc} = a^{(b^c)}$ (for pairs (b, c) where b^c is defined). Because $a > 0$, both sides are positive, and we can take logarithms (Theorem 0.4):

$$\begin{aligned} a^{bc} &= a^{(b^c)} \\ \log\{a^{bc}\} &= \log\{a^{(b^c)}\} \quad (\text{take logarithms of both sides}) \\ bc \cdot \log\{a\} &= b^c \cdot \log\{a\} \quad (\text{logarithm of a power}) \end{aligned} \tag{2}$$

The last line of Equation 2 holds exactly when

1. $a = 1$ (so that $\log\{a\} = 0$), or
2. $bc = b^c$ (see Theorem 0.10).

Case $a = 0$. Here we cannot take logarithms, so we work from the values of powers of 0: $0^s = 0$ for $s > 0$ (Definition 0.26), $0^0 = 1$, and 0^s is undefined for $s < 0$.

- If $b < 0$, then 0^b is undefined, so $(a^b)^c$ is undefined.
- If $b > 0$, then $(0^b)^c = 0^c$ and $b^c > 0$, so $0^{(b^c)} = 0$; the two sides agree exactly when $c > 0$.
- If $b = 0$, then

$$\begin{aligned}(0^0)^c &= 1^c \\ &= 1;\end{aligned}$$

for $c > 0$,

$$\begin{aligned}0^{(0^c)} &= 0^0 \\ &= 1,\end{aligned}$$

so the two sides agree, and for $c \leq 0$ they do not.

So for $a = 0$, $(a^b)^c = a^{(b^c)}$ exactly when $b \geq 0$ and $c > 0$.

In particular, when $a = 0$, $b \geq 0$, and $c = 0$, the two sides differ:

$$\begin{aligned}(a^b)^c &= (0^b)^0 \\ &= 1 \\ a^{(b^c)} &= 0^{(b^0)} \\ &= 0^1 \\ &= 0\end{aligned}$$

Theorem 0.11 (When repeated exponentiation is not reordered). *Let $a \geq 0$ and $b, c \in \mathbb{R}$, where b^c is defined. Then $(a^b)^c = a^{(b^c)}$, with both sides defined, if and only if*

- $a > 0$ and either $a = 1$ or $bc = b^c$; or
- $a = 0$, $b \geq 0$, and $c > 0$.

Proof

Proof. This is the solution to Exercise 0.3, worked in Solution 0.3. □

- [1] Wikipedia contributors, “Polynomial — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Polynomial>
- [2] Wikipedia contributors, “Quadratic function — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Quadratic_function
- [3] Wikipedia contributors, “Parabola — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Parabola>
- [4] Wikipedia contributors, “Monomial — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Monomial>
- [5] Wikipedia contributors, “Affine transformation — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Affine_transformation

- [6] Wikipedia contributors, “Slope — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Slope>
- [7] Wikipedia contributors, “Linear function — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Linear_function
- [8] Wikipedia contributors, “Limit of a sequence — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Limit_of_a_sequence
- [9] Wikipedia contributors, “Series (mathematics) — Wikipedia, the free encyclopedia.” 2026. Available: [https://en.wikipedia.org/wiki/Series_\(mathematics\)](https://en.wikipedia.org/wiki/Series_(mathematics))
- [10] Wikipedia contributors, “Monotone convergence theorem — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Monotone_convergence_theorem
- [11] Wikipedia contributors, “Exponential function — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Exponential_function
- [12] Wikipedia contributors, “E (mathematical constant) — Wikipedia, the free encyclopedia.” 2026. Available: [https://en.wikipedia.org/wiki/E_\(mathematical_constant\)](https://en.wikipedia.org/wiki/E_(mathematical_constant))
- [13] Wikipedia contributors, “Natural logarithm — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Natural_logarithm
- [14] Wikipedia contributors, “Logarithm — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Logarithm>
- [15] Wikipedia contributors, “Tarski’s high school algebra problem — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Tarski%27s_high_school_algebra_problem
- [16] Wikipedia contributors, “Contour line — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Contour_line