

Complex Numbers

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Complex numbers

Definition 0.1 (Imaginary unit). The **imaginary unit** i is a number whose square is -1 [1]:

$$i^2 \stackrel{\text{def}}{=} -1.$$

Apart from that one rule, i obeys the usual rules of arithmetic:

- sums and products with i are [commutative](#);
- sums and products with i are [associative](#);
- multiplication [distributes](#) over addition.

Axler [2], Definition 1.1, p. 2 makes this idea rigorous: it defines a complex number as an ordered pair (a, b) of real numbers, written $a + bi$, defines addition and multiplication of such pairs, writes $0 + 1i$ as i , and leaves it to the reader to verify that $i^2 = -1$.

Example 0.1 (Powers of the imaginary unit). The powers of i repeat in a cycle of four:

$$\begin{aligned} i^3 &= i^2 \cdot i && \text{(split off one factor of } i\text{)} \\ &= (-1) \cdot i && \text{(Definition 0.1)} \\ &= -i && \text{(multiply)} \end{aligned}$$

and

$$\begin{aligned} i^4 &= i^2 \cdot i^2 && \text{(split the power into two squares)} \\ &= (-1) \cdot (-1) && \text{(Definition 0.1, twice)} \\ &= 1 && \text{(multiply)} \end{aligned}$$

so the cycle starts again:

$$\begin{aligned}
 i^5 &= i^4 \cdot i \quad (\text{split off one factor of } i) \\
 &= 1 \cdot i \quad (i^4 = 1) \\
 &= i \quad (\text{multiply})
 \end{aligned}$$

and the powers run $i, -1, -i, 1, i, \dots$

Example 0.2 (No real number squares to -1). The imaginary unit is not a real number, because the square of every real number x is at least 0:

- if $x \geq 0$, then $x^2 = x \cdot x$ is a product of two nonnegative numbers, so $x^2 \geq 0$;
- if $x < 0$, then $x^2 = x \cdot x$ is a product of two negative numbers, so $x^2 > 0$.

For instance, $3^2 = 9$ and $(-3)^2 = 9$; neither is -1 . So the equation $x^2 = -1$ has no real solution, and Definition 0.1 introduces a new number to solve it.

Theorem 0.1 (The number $-i$ also squares to -1).

$$(-i)^2 = -1.$$

Proof

Proof.

$$\begin{aligned}
 (-i)^2 &= (-1)^2 i^2 \quad (-i = (-1) \cdot i, \text{ and } (ab)^2 = a^2 b^2 \text{ because products commute and associate}) \\
 &= 1 \cdot (-1) \quad ((-1)^2 = 1 \text{ and Definition 0.1}) \\
 &= -1 \quad (\text{multiply})
 \end{aligned}$$

□

Example 0.3 (Squaring i and $-i$ in R). R has the imaginary unit as `1i`. The code below squares it and its negative.

```
squares <- c(i = (1i)^2, minus_i = (-1i)^2)
squares
#>      i minus_i
#> -1+0i -1+0i
```

Both squares have real part -1 and imaginary part 0, as Definition 0.1 and Theorem 0.1 say.

i Remark (The rule does not tell i from $-i$)

Remark 0.1 (The rule does not tell i from $-i$). Because $-i$ obeys the same rule as i (Theorem 0.1), the rule $i^2 = -1$ does not single one of them out. The symbol i names one of these two square roots of -1 , chosen once and for all.

Definition 0.2 (Complex number). A **complex number** [3] is a number of the form

$$z = a + b i,$$

where a and b are real numbers and i is the imaginary unit (Definition 0.1). The set of all complex numbers is $\mathbb{C} \stackrel{\text{def}}{=} \{a + b i : a, b \in \mathbb{R}\}$.

Definition 0.3 (Real part). The **real part** of a complex number $z = a + b i$ (Definition 0.2) is $\text{Re } z \stackrel{\text{def}}{=} a$.

Definition 0.4 (Imaginary part). The **imaginary part** of a complex number $z = a + b i$ (Definition 0.2) is $\text{Im } z \stackrel{\text{def}}{=} b$.

Definition 0.5 (Equal complex numbers). Two complex numbers (Definition 0.2) are **equal** exactly when their real parts (Definition 0.3) are equal and their imaginary parts (Definition 0.4) are equal.

Axler [2], Definition 1.1, p. 2 defines $\mathbb{C}$ as the set of ordered pairs (a, b) of real numbers, written $a + b i$, and Axler [2], Definition 4.1, p. 120 defines the real and imaginary parts.

Example 0.4 (Real and imaginary parts). For $z = 2 - 5 i$, the real part is $\text{Re } z = 2$ and the imaginary part is $\text{Im } z = -5$: the imaginary part is the real number -5 , not $-5 i$. The numbers $1 + i$ and $1 - i$ have the same real part, 1, but different imaginary parts, 1 and -1 , so they are different complex numbers. The real number 7 is the complex number $7 + 0 i$, with real part 7 and imaginary part 0.

Theorem 0.2 (Every real number is a complex number). *A complex number z is a real number exactly when $\text{Im } z = 0$. In particular, every real number a is the complex number $a + 0 i$, with $\text{Re } a = a$ and $\text{Im } a = 0$, so $\mathbb{R}$ is a subset of $\mathbb{C}$.*

i Proof

Proof. Let a be a real number. Since $0i = 0$,

$$\begin{aligned} a + 0i &= a + 0 \quad (0i = 0) \\ &= a \quad (\text{Theorem 4 in Convexity, Infimum and Sums}) \end{aligned}$$

so a is the complex number $a + 0i$, whose imaginary part is 0. The real part of $a + 0i$ is a , and by the equality rule in Definition 0.5 no other pair of real and imaginary parts gives the same number.

Conversely, if $z = a + bi$ has $b = \text{Im } z = 0$, then $z = a + 0i = a$ by the same calculation, so z is real. $\square$

Example 0.5 (Which complex numbers are real). R stores complex numbers as a real part and an imaginary part. The code below converts the real number 7 to a complex number and reads the parts of three complex numbers.

```
z <- c(as.complex(7), 2 - 5i, 3 + 0i)
data.frame(z = z, re = Re(z), im = Im(z), is_real = Im(z) == 0)
#>      z re im is_real
#> 1 7+0i  7  0    TRUE
#> 2 2-5i  2 -5   FALSE
#> 3 3+0i  3  0    TRUE
```

The real number 7 becomes the complex number $7+0i$, with imaginary part 0. Of the 3 numbers, 2 have imaginary part 0, and by Theorem 0.2 these are the real ones. The number $2 - 5i$ is not real, because its imaginary part is -5.

Theorem 0.3 (Adding and multiplying complex numbers). *For real numbers a, b, c, d :*

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

i Proof

Proof. **Sum.**

$$\begin{aligned}
 (a + bi) + (c + di) &= a + c + bi + di && \text{(rearrange the terms)} \\
 &= (a + c) + (b + d)i && \text{(factor out } i)
 \end{aligned}$$

Product.

$$\begin{aligned}
 (a + bi)(c + di) &= ac + adi + bci + bdi^2 && \text{(distribute)} \\
 &= ac + adi + bci - bd && \text{(Definition 0.1)} \\
 &= (ac - bd) + (ad + bc)i && \text{(group the real terms and the terms with } i)
 \end{aligned}$$

□

Axler [2], Definition 1.1, p. 2 takes these two formulas as the definitions of addition and multiplication.

Example 0.6 (Adding and multiplying two complex numbers). Let $w = 1 + 2i$ and $z = 3 - i$. Their sum is

$$\begin{aligned}
 w + z &= (1 + 3) + (2 + (-1))i && \text{(Theorem 0.3, sum)} \\
 &= 4 + i && \text{(add)}
 \end{aligned}$$

and their product, multiplying out directly, is

$$\begin{aligned}
 wz &= 1 \cdot 3 + 1 \cdot (-i) + 2i \cdot 3 + 2i \cdot (-i) && \text{(distribute)} \\
 &= 3 - i + 6i - 2i^2 && \text{(multiply)} \\
 &= 3 - i + 6i + 2 && \text{(Definition 0.1)} \\
 &= (3 + 2) + (-1 + 6)i && \text{(group the real terms and the terms with } i) \\
 &= 5 + 5i && \text{(add)}
 \end{aligned}$$

The product formula in Theorem 0.3 gives the same answer: with $a = 1$, $b = 2$, $c = 3$ and $d = -1$,

$$\begin{aligned}
 ac - bd &= 3 - (-2) \\
 &= 5
 \end{aligned}$$

and

$$\begin{aligned}
 ad + bc &= -1 + 6 \\
 &= 5.
 \end{aligned}$$

Definition 0.6 (Complex conjugate). The **complex conjugate** [4] of a complex number $z = a + bi$ (Definition 0.2) is

$$\bar{z} \stackrel{\text{def}}{=} a - bi.$$

Axler [2], Definition 4.2, p. 120 gives the same definition, as $\bar{z} = \operatorname{Re} z - (\operatorname{Im} z) i$.

Example 0.7 (Complex conjugates).

- $\overline{3 + 4i} = 3 - 4i.$

-

$$\begin{aligned} \overline{-2i} &= \overline{0 + (-2)i} \\ &= 0 - (-2)i \\ &= 2i. \end{aligned}$$

-

$$\begin{aligned} \bar{5} &= \overline{5 + 0i} \\ &= 5 - 0i \\ &= 5 : \end{aligned}$$

a real number is its own complex conjugate. A complex number with a nonzero imaginary part, such as $3 + 4i$, is not.

Definition 0.7 (Absolute value of a complex number, modulus). The **absolute value**, or **modulus** [3], of a complex number $z = a + bi$ (Definition 0.2) is

$$|z| \stackrel{\text{def}}{=} \sqrt{a^2 + b^2}.$$

Axler [2], Definition 4.2, p. 120 gives the same definition.

Example 0.8 (Absolute values of complex numbers).

-

$$\begin{aligned} |3 + 4i| &= \sqrt{3^2 + 4^2} \\ &= \sqrt{25} \\ &= 5. \end{aligned}$$

-

$$\begin{aligned} |-2i| &= \sqrt{0^2 + (-2)^2} \\ &= \sqrt{4} \\ &= 2. \end{aligned}$$

- For a real number $a = a + 0i$,

$$\begin{aligned}|a| &= \sqrt{a^2 + 0^2} \\ &= \sqrt{a^2},\end{aligned}$$

which is the usual absolute value of a ; for instance,

$$\begin{aligned}|-3| &= \sqrt{9} \\ &= 3.\end{aligned}$$

Theorem 0.4 (A complex number times its conjugate). *For every complex number z ,*

$$z \bar{z} = |z|^2.$$

In particular, $z \bar{z}$ is a nonnegative real number.

i Proof

Proof. Write $z = a + bi$ with a, b real. Then

$$\begin{aligned}z \bar{z} &= (a + bi)(a - bi) && \text{(Definition 0.6)} \\ &= a^2 - abi + abi - b^2 i^2 && \text{(distribute)} \\ &= a^2 - b^2 i^2 && \text{(cancel } abi\text{)} \\ &= a^2 + b^2 && \text{(Definition 0.1)} \\ &= |z|^2 && \text{(Definition 0.7)}\end{aligned}$$

□

Axler [2], result 4.4, p. 121 lists this identity among the properties of complex numbers.

Example 0.9 (Multiplying $3 + 4i$ by its conjugate).

$$\begin{aligned}(3 + 4i)(3 - 4i) & \\ &= 9 - 12i + 12i - 16i^2 && \text{(distribute)} \\ &= 9 - 16i^2 && \text{(cancel } 12i\text{)} \\ &= 9 + 16 && \text{(Definition 0.1)} \\ &= 25 && \text{(add)}\end{aligned}$$

which is $|3 + 4i|^2 = 5^2$ from Example 0.8, as Theorem 0.4 says.

- [1] Wikipedia contributors, “Imaginary unit — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Imaginary_unit
- [2] S. Axler, *Linear algebra done right*, 4th ed. in Undergraduate texts in mathematics. Springer, 2024. doi: [10.1007/978-3-031-41026-0](https://doi.org/10.1007/978-3-031-41026-0).
- [3] Wikipedia contributors, “Complex number — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Complex_number
- [4] Wikipedia contributors, “Complex conjugate — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Complex_conjugate