

Equalities, Inequalities and Minimizers

Last modified: 2026-10-27 20:53:54 (PDT)

Equalities

Mastery of [Elementary Algebra](#) (a.k.a. “College Algebra”) is a prerequisite for calculus, which is in turn a prerequisite for most statistics and data science courses. Nevertheless, each year, some students are still uncomfortable with algebraic manipulations of mathematical formulas. Therefore, I include this section as a quick reference.

Definition 0.1 (Transitive relation). Let $\sim$ be a [relation](#) on a set S . The relation $\sim$ is **transitive** [1] if, for all $a, b, c \in S$, $a \sim b$ and $b \sim c$ together imply $a \sim c$.

Example 0.1 (A transitive relation and one that is not).

- $\leq$ is transitive: for example, $1 \leq 2$ and $2 \leq 5$, and indeed $1 \leq 5$.
- $\neq$ is not transitive: $1 \neq 2$ and $2 \neq 1$, but $1 \neq 1$ is false.

Theorem 0.1 (Equalities are transitive). *If $a = b$ and $b = c$, then $a = c$. That is, $=$ is a [transitive](#) relation.*

Theorem 0.2 (Substituting equivalent expressions). *If $a = b$, then for any function $f(x)$, $f(a) = f(b)$*

Definition 0.2 (Identity). An **identity** [2] is an equation between two expressions that holds for every value of its variables in a stated set, such as the [real numbers](#).

Example 0.2 (An identity, and an equation that is not one).

- $2(x + 1) = 2x + 2$ is an identity: it holds for every real x . For example, at $x = 3$, both sides equal 8.

- $x + 1 = 3$ is not an identity: it holds at $x = 2$, but at $x = 0$ the left side is 1 and the right side is 3.

Inequalities

Definition 0.3 (Strict inequalities). For **real numbers** a and b , $a < b$ (“ a is less than b ”) and $a > b$ (“ a is greater than b ”) are **strict inequalities** [3].

Definition 0.4 (Non-strict inequalities). For **real numbers** a and b , the **non-strict inequalities** also allow equality, unlike the strict inequalities (Definition 0.3):

- $a \leq b$ means $a < b$ or $a = b$;
- $a \geq b$ means $a > b$ or $a = b$.

Example 0.3 (Strict and non-strict inequalities).

- $3 \leq 3$ is true, because $3 = 3$, but $3 < 3$ is false.
- $2 < 3$ and $2 \leq 3$ are both true.
- $4 \geq 5$ is false, because $4 > 5$ and $4 = 5$ are both false.

Theorem 0.3 (Adding to both sides of an inequality). *If $a < b$, then $a + c < b + c$*

Theorem 0.4 (Negating both sides of an inequality). *If $a < b$, then: $-a > -b$*

Theorem 0.5 (Multiplying both sides of an inequality by a positive number). *If $a < b$ and $c > 0$, then $ca < cb$.*

Theorem 0.6 (Negation is multiplication by -1).

$$-a = (-1) * a$$

Powers, square roots, and absolute values

Definition 0.5 (Power). Let a be a real number and n a [natural number](#). The n th **power** of a is the product of n copies of a [4]:

$$a^n \stackrel{\text{def}}{=} \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

Definition 0.6 (Base of a power). In a^n (Definition [0.5](#)), a is the **base**.

Definition 0.7 (Exponent of a power). In a^n (Definition [0.5](#)), n is the **exponent**.

Definition 0.8 (Square). The second power a^2 (Definition [0.5](#)) is the **square** of a .

Example 0.4 (Powers).

•

$$\begin{aligned} 2^3 &= 2 \cdot 2 \cdot 2 \\ &= 8, \end{aligned}$$

with base 2 and exponent 3.

•

$$\begin{aligned} (-3)^2 &= (-3) \cdot (-3) \\ &= 9, \end{aligned}$$

but

$$\begin{aligned} -3^2 &= -(3^2) \\ &= -9 : \end{aligned}$$

the exponent applies only to the 3.

• $5^0 = 1$.

•

$$\begin{aligned} 2^{-2} &= \frac{1}{2^2} \\ &= \frac{1}{4}. \end{aligned}$$

Definition 0.9 (Powers with exponent zero or a negative integer). For the exponent 0 and negative [integer](#) exponents:

- $a^0 \stackrel{\text{def}}{=} 1$, including $0^0 = 1$ by convention;
- $a^{-n} \stackrel{\text{def}}{=} \frac{1}{a^n}$, when $a \neq 0$.

i Remark (Powers with real exponents)

Remark 0.1 (Powers with real exponents). [Definition 26 in Polynomials, Exponentials and Logarithms](#) extends powers to real exponents when the base is positive.

Definition 0.10 (Square root). For a real number $a \geq 0$, the **square root** of a , written $\sqrt{a}$ [5], is the non-negative real number whose square (Definition [0.5](#)) is a .

Example 0.5 (Square roots).

- $\sqrt{9} = 3$. Although $(-3)^2 = 9$ too, -3 is negative, so $\sqrt{9}$ is not -3 .
- $\sqrt{0} = 0$.
- $\sqrt{2} \approx 1.41421$.
- $\sqrt{-4}$ is not a real number, because no real number squares to -4 (see [Section 1 in Complex Numbers](#)).

Definition 0.11 (Absolute value of a real number). The **absolute value** of a real number a [6] is

$$|a| \stackrel{\text{def}}{=} \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

Example 0.6 (Absolute values).

- $|2.5| = 2.5$, because $2.5 \geq 0$.

-

$$\begin{aligned} |-3| &= -(-3) \\ &= 3, \end{aligned}$$

because $-3 < 0$.

- $|0| = 0$.

•

$$\begin{aligned} |-3| &= 3 \\ &= \sqrt{9} \\ &= \sqrt{(-3)^2} : \end{aligned}$$

for every real a , $|a| = \sqrt{a^2}$ (Definition 0.10).

- $|x - 1| < 1$ says that x is less than 1 away from 1, that is, $0 < x < 2$.

i Remark (Absolute value as distance)

Remark 0.2 (Absolute value as distance). For real numbers a and b , $|a - b|$ is the distance between a and b on the number line.

Percent

The word “percent” comes from the Latin “per centum”; “centum” is Latin for 100, so “percent” means “per hundred” [7].

Definition 0.12 (Percent). For a real number x , “ x percent”, written $x\%$, means $x/100$ [7]:

$$x\% \stackrel{\text{def}}{=} \frac{x}{100}$$

The percent sign “%” is a shorthand for “/100”.

Example 0.7 (Ten percent is one tenth). Contrary to what you may have learned previously, the equality $10\% = 0.1$ is true and correct, just as $10\text{kg} = 10,000\text{g}$ is true and correct:

$$\begin{aligned} 10\% &= \frac{10}{100} \quad (\text{definition of percent}) \\ &= 0.1 \quad (\text{divide}) \end{aligned}$$

You are welcome to switch between decimal and percent notation freely; just make sure you execute it correctly.

Minimum, maximum, argmin and argmax

Exercise 0.1 (Smallest value, and where it occurs).

1. Let $f(x) = (x - 2)^2 + 1$ for every $x \in \mathbb{R}$. What is the smallest value f takes, and at which x does it take that value?
2. Now let $g(x) = (x - 2)^2 + 1$ only for $x \in \{0, 1, 3, 4\}$. What is the smallest value g takes, and at which x does it take that value?

Solution

Solution 0.1.

1. The square of a real number is never negative, so $(x - 2)^2 \geq 0$ for every x , and therefore $f(x) \geq 0 + 1 = 1$. The square is 0 only when $x - 2 = 0$, that is, when $x = 2$, and there $f(2) = 1$. So the smallest value of f is 1, and $x = 2$ is the only input where f takes it.
2. Evaluate g at each of its four inputs:

x	0	1	3	4
$g(x)$	5	2	2	5

The smallest value of g is 2, and g takes it at two inputs, $x = 1$ and $x = 3$. The answer to “where?” is the set $\{1, 3\}$, not a single number.

Definition 0.13 (Minimum). Let $A \subseteq \mathbb{R}$. A number m is the **minimum** of A [8], written $\min A$, if $m \in A$ and $m \leq a$ for all $a \in A$.

i Remark (Not every set has a minimum)

Remark 0.3 (Not every set has a minimum). The set $\{3, 1, 4\}$ has minimum 1, and the [interval](#) $[0, 1]$ has minimum 0. The interval $(0, 1]$ has no minimum: every element a of it has a smaller element, $a/2$, also in it.

Definition 0.14 (Maximum). Let $A \subseteq \mathbb{R}$. A number M is the **maximum** of A [8], written $\max A$, if $M \in A$ and $a \leq M$ for all $a \in A$.

Example 0.8 (A maximum, and a set without one).

- $\max\{-2, 5, 0\} = 5$: 5 is in the set, and $-2 \leq 5$, $5 \leq 5$, $0 \leq 5$.
- The interval $[0, 1)$ has no maximum. For any $a \in [0, 1)$, the number $\frac{a+1}{2}$ is also in $[0, 1)$, and it is larger than a , since $\frac{a+1}{2} > a$ exactly when $a < 1$. So no element of $[0, 1)$ is at least as large as every element.

Definition 0.15 (Argmin). Let $f : A \rightarrow \mathbb{R}$ be a [function](#). The **argmin** of f over A [9] is the set of inputs where f takes its smallest value:

$$\arg \min_{x \in A} f(x) \stackrel{\text{def}}{=} \{x \in A : \forall a \in A, f(x) \leq f(a)\}$$

When this set has exactly one element $\hat{x}$, we write $\hat{x} = \arg \min_{x \in A} f(x)$.

i Remark (Smallest value versus where it occurs)

Remark 0.4 (Smallest value versus where it occurs). The smallest value itself is $\min f(A)$, the minimum (Definition 0.13) of the [image](#) of f , and the argmin is where that value is attained. For example, let $f(x) = x^2$ on $A = \{-1, 0, 2\}$. The image is $f(A) = \{1, 0, 4\}$, so the smallest value is $\min f(A) = 0$, and $\arg \min_{x \in A} f(x) = \{0\}$.

The argmin is empty when f has no smallest value, for example $f(x) = x$ on $(0, 1]$.

Definition 0.16 (Argmax). Let $f : A \rightarrow \mathbb{R}$ be a [function](#). The **argmax** of f over A [9] is the set of inputs where f takes its largest value:

$$\arg \max_{x \in A} f(x) \stackrel{\text{def}}{=} \{x \in A : \forall a \in A, f(a) \leq f(x)\}$$

When this set has exactly one element $\hat{x}$, we write $\hat{x} = \arg \max_{x \in A} f(x)$.

Example 0.9 (An argmax with two points, and an empty one).

- Let $g(x) = x^2$ on $A = \{-2, 0, 1, 2\}$. The values are $g(-2) = 4$, $g(0) = 0$, $g(1) = 1$ and $g(2) = 4$, so the largest value is 4 and $\arg \max_{x \in A} g(x) = \{-2, 2\}$. This argmax has two elements, so it is not written as a single $\hat{x}$.
- Let $f(x) = x$ on $[0, 1)$. As in Example 0.8, for every input x the input $\frac{x+1}{2}$ has a larger value, so no input attains a largest value, and $\arg \max_{x \in [0, 1)} f(x)$ is the [empty set](#).

Definition 0.17 (Positive part of a function). Let S be a [set](#), and let $f : S \rightarrow \mathbb{R}$ be a [function](#). The **positive part** of f is the non-negative function

$$f^+(s) \stackrel{\text{def}}{=} \max\{f(s), 0\}.$$

Example 0.10 (The positive part of the identity function). For $f(x) = x$ on $\mathbb{R}$, the [positive part](#) is $f^+(x) = \max\{x, 0\}$. At $x = 3$, $f^+(3) = \max\{3, 0\} = 3$, and at $x = -2$, $f^+(-2) = \max\{-2, 0\} = 0$.

Definition 0.18 (Negative part of a function). For f as in Definition [0.17](#), the **negative part** of f is the non-negative [function](#)

$$f^-(s) \stackrel{\text{def}}{=} \max\{-f(s), 0\}.$$

Example 0.11 (The negative part of the identity function). For $f(x) = x$ on $\mathbb{R}$, the [negative part](#) is $f^-(x) = \max\{-x, 0\}$. At $x = 3$, $f^-(3) = \max\{-3, 0\} = 0$, and at $x = -2$, $f^-(-2) = \max\{2, 0\} = 2$.

Global and local minimizers

Exercise 0.2 (A dip that is not the bottom). Let $f(x) = x^3 - 3x$ for $x \in \mathbb{R}$.

1. Factor $f(x) - f(1)$ completely.
2. Use part 1 to compare $f(x)$ with $f(1)$ for every x in the interval $(0, 2)$. Is $f(x)$ larger than, smaller than, or equal to $f(1)$, and for which x ?
3. Compute $f(-3)$. Is $f(1)$ the smallest value f takes on all of $\mathbb{R}$?
4. Does f take a smallest value anywhere on $\mathbb{R}$?

Solution

Solution 0.2.

1. Since

$$\begin{aligned} f(1) &= 1 - 3 \\ &= -2, \end{aligned}$$

so

$$\begin{aligned}
f(x) - f(1) &= x^3 - 3x + 2 \\
&= (x - 1)(x^2 + x - 2) \\
&= (x - 1)(x - 1)(x + 2) \\
&= (x - 1)^2(x + 2).
\end{aligned}$$

2. For $x \in (0, 2)$, $(x - 1)^2 \geq 0$ and $x + 2 > 0$, so their product is at least 0. By part 1, $f(x) - f(1) \geq 0$, that is, $f(x) \geq f(1)$. So $f(x)$ is never smaller than $f(1)$ on $(0, 2)$. It equals $f(1)$ only when $(x - 1)^2 = 0$, that is, only at $x = 1$, and it is larger than $f(1)$ at every other $x \in (0, 2)$.

3.

$$\begin{aligned}
f(-3) &= -27 + 9 \\
&= -18.
\end{aligned}$$

Since $-18 < -2 = f(1)$, $f(1)$ is not the smallest value of f on $\mathbb{R}$.

4. No. For each $n \geq 2$,

$$\begin{aligned}
f(-n) &= -n^3 + 3n \\
&= -n(n^2 - 3) \\
&\leq -n,
\end{aligned}$$

so no value of f is smaller than all the others: for any x , picking $n \geq 2$ with $-n < f(x)$ gives $f(-n) < f(x)$.

Definition 0.19 (Global minimizer). Let $A \subseteq \mathbb{R}^p$ and let $f : A \rightarrow \mathbb{R}$ be a [function](#). A point $x^* \in A$ is a **global minimizer** of f over A if $f(x^*) \leq f(x)$ for all $x \in A$.

i Remark (Global minimizers form the argmin)

Remark 0.5 (Global minimizers form the argmin). The global minimizers of f are exactly the elements of $\arg \min_{x \in A} f(x)$ (Definition [0.15](#)). A function can have more than one global minimizer. For example, $f(x) = (x^2 - 1)^2$ on $\mathbb{R}$ satisfies $f(x) \geq 0$ for every x , and $f(x) = 0$ exactly when $x = -1$ or $x = 1$, so its global minimizers are -1 and 1 , and $\arg \min_{x \in \mathbb{R}} f(x) = \{-1, 1\}$.

Definition 0.20 (Local minimizer). Let $A \subseteq \mathbb{R}^p$ and let $f : A \rightarrow \mathbb{R}$ be a [function](#). A point $x^* \in A$ is a **local minimizer** of f if there is a number $\delta > 0$ such that $f(x^*) \leq f(x)$ for all $x \in A$ with $\|x - x^*\| < \delta$, where $\|\cdot\|$ is the [Euclidean norm](#).

i Remark (Local and global minimizers)

Remark 0.6 (Local and global minimizers). For $p = 1$, $\|x - x^*\| = |x - x^*|$. For example, with $x^* = 1$ and $\delta = 1$, the condition $|x - 1| < 1$ means $0 < x < 2$.

Every global minimizer (Definition 0.19) is a local minimizer: take any $\delta > 0$. For example, $x^* = 0$ is a global minimizer of $f(x) = x^2$ on $\mathbb{R}$, so it is also a local minimizer.

The **converse** fails. In Exercise 0.2, $x^* = 1$ is a local minimizer of $f(x) = x^3 - 3x$ (take $\delta = 1$), but not a global one.

Example 0.12 (A point that is not a local minimizer). For $f(x) = x^3 - 3x$, the point $x = -1$ is not a local minimizer. For any t ,

$$\begin{aligned} f(-1+t) &= (-1+t)^3 - 3(-1+t) && \text{(substitute)} \\ &= (-1+3t-3t^2+t^3) - 3(-1+t) && \text{(expand the cube)} \\ &= -1+3t-3t^2+t^3+3-3t && \text{(multiply } -3 \text{ into the parentheses)} \\ &= 2-3t^2+t^3 && (3t-3t=0 \text{ and } -1+3=2) \\ &= 2+t^2(t-3) && \text{(factor out } t^2) \\ &= f(-1)+t^2(t-3), && (f(-1)=(-1)^3-3(-1), \text{ which equals } 2) \end{aligned}$$

and $t^2(t-3) < 0$ whenever $t \neq 0$ and $t < 3$. So whatever $\delta > 0$ is, the point $x = -1+t$ with $t = \min\{\delta/2, 1\}$ satisfies $|x - (-1)| < \delta$ and $f(x) < f(-1)$. For example,

$$\begin{aligned} f(-0.9) &= -0.729 + 2.7 \\ &= 1.971 \\ &< 2. \end{aligned}$$

- [1] Wikipedia contributors, “Transitive relation — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Transitive_relation
- [2] Wikipedia contributors, “Identity (mathematics) — Wikipedia, the free encyclopedia.” 2026. Available: [https://en.wikipedia.org/wiki/Identity_\(mathematics\)](https://en.wikipedia.org/wiki/Identity_(mathematics))
- [3] Wikipedia contributors, “Inequality (mathematics) — Wikipedia, the free encyclopedia.” 2026. Available: [https://en.wikipedia.org/wiki/Inequality_\(mathematics\)](https://en.wikipedia.org/wiki/Inequality_(mathematics))
- [4] Wikipedia contributors, “Exponentiation — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Exponentiation>
- [5] Wikipedia contributors, “Square root — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Square_root
- [6] Wikipedia contributors, “Absolute value — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Absolute_value

- [7] Wikipedia contributors, “Percentage — Wikipedia, the free encyclopedia.” 2026. Available: <https://en.wikipedia.org/wiki/Percentage>
- [8] Wikipedia contributors, “Maxima and minima — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Maxima_and_minima
- [9] Wikipedia contributors, “Arg max — Wikipedia, the free encyclopedia.” 2026. Available: https://en.wikipedia.org/wiki/Arg_max