Sets

Elements, subsets, products and operations

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Source

Adapted from Kameron Decker Harris's Fall 2026 lecture notes "Sets 0" to "Sets 3", which are typed transcriptions of Scott Wehrwein's Spring 2025 handwritten lecture notes. These come from those notes:

- roster notation, and that order and repetition do not matter in a set;
- the table of sets of numbers;
- set-builder notation with a test and with an expression;
- cardinality, and the pattern of the example of a set whose elements are sets;
- countably infinite sets, and the diagonal argument that the real numbers are not countable;
- subsets, strict subsets and power sets;
- tuples and Cartesian products, including products of three sets;
- union, intersection, difference, complement and the universal set (with its example of the even integers), with their other notations;
- Venn diagrams.

The exercises are new.

Sets and their elements

Exercise 1 (Days that begin with T)**.** Let W be the set of days of the week whose English names begin with the letter T. List the elements of W.

Solution

Solution 1. $W = \{\text{Tuesday}, \text{Thursday}\}$.

Definition 1 (Set)**.** A **set** is a collection of distinct objects, called its elements. We write $a \in A$ when a is an element of the set A, and $a \notin A$ when it is not. Two sets are equal when they have exactly the same elements.

Remark 1 (Naive and formal set theory). Definition 1 is the informal ("naive") notion of a set, which is all these notes need. Formal set theory states axioms for sets instead (Wikipedia contributors 2026n). For example, the last sentence of Definition 1, that two sets are equal when they have exactly the same elements, is one of those axioms, the *axiom of extensionality*.

Example 1 (Sets of die rolls)**.** The possible results of rolling a six-sided die form the set $\{1, 2, 3, 4, 5, 6\}$, with $3 \in \{1, 2, 3, 4, 5, 6\}$ and $7 \notin \{1, 2, 3, 4, 5, 6\}$. The sets $\{1, 2, 3\}$ and $\{3, 2, 1\}$ are equal, because they have the same elements; the order in which the elements are listed does not matter.

Definition 2 (Roster notation)**.** Writing a set by listing its elements between braces, as in $\{1, 2, 3\}$, is called **roster notation** (Wikipedia contributors 2026n). An ellipsis, "...", can stand for elements that follow a clear pattern, as in $\{10, 20, 30, \dots, 90\}$.

Example 2 (Order and repetition do not matter)**.** The sets $\{3, 1, 2\}$, $\{1, 2, 3\}$ and $\{1, 1, 2, 3, 3\}$ are equal, because each has exactly the elements 1, 2 and 3 (Definition 1). A roster can list any objects, not only numbers: $\{\text{red}, \text{green}, \text{blue}\}$ is a set with three elements.

Definition 3 (Natural numbers (our convention))**.** In these notes, the **natural numbers** (Wikipedia contributors 2026j) are the positive integers (Definition: Positive integers):

$$\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$$

Other conventions for the natural numbers

Other sources may define $\mathbb{N}$ differently, so check each source's definition before reading its formulas:

- Many sources include 0 in $\mathbb{N}$. The international standard ISO 80000-2 defines $\mathbb{N}$ to include 0, continuing the earlier standard ISO 31-11 (1978).
- Other sources start $\mathbb{N}$ at 1, as these notes do.
- To remove the ambiguity, some sources write $\mathbb{N}_1$ or $\mathbb{Z}^+$ for $\{1, 2, 3, ...\}$, and $\mathbb{N}_0$ or $\mathbb{Z}^{0+}$ for $\{0, 1, 2, ...\}$.
- The words vary too. "Positive integers" ($\{1, 2, 3, ...\}$) and "non-negative integers" ($\{0, 1, 2, ...\}$) are unambiguous. "Whole numbers" usually includes 0, but can also mean all of the integers, negative ones included. "Counting numbers" usually starts at 1, but some sources include 0.
- Some older texts write J for the natural numbers.

When a formula's meaning depends on whether 0 is included, we write the set out explicitly, for example $\{0, 1, 2, ...\}$ or $\{1, \dots, n\}$.
Source: Wikipedia, "Natural number", "Terminology and notation" and "Zero as natural number", which cites ISO 80000-2:2019 and the textbooks using each notation.

Definition 4 (Real numbers)**.** A **real number** (Wikipedia contributors 2026m) is a number that can be written as a decimal expansion: an integer part, followed by a decimal point and a finite or infinite list of decimal places after it. Equivalently, the real numbers are the points on the number line. The set of real numbers is written $\mathbb{R}$.

Remark 2 (A working definition)*.* Definition 4 is a working definition, not a construction: it does not say what an infinite decimal expansion is, or how to add or multiply two of them. Rudin (1976, Appendix to Chapter 1) constructs the real numbers rigorously. Some real numbers have two decimal expansions: for example, $0.999\ldots$ and $1.000\ldots$ are the same real number, 1.

Remark 3 (Sets of numbers). These notes use these sets of numbers:

Symbol	Name	Elements
$\mathbb{N}$	natural numbers (Definition 3)	$1, 2, 3, ...$
$\mathbb{Z}$	integers (Definition: Integers)	$..., -2, -1, 0, 1, 2, ...$
$\mathbb{Q}$	rational numbers (Definition: Rational numbers)	fractions $\frac{a}{b}$ of integers with $b \neq 0$
$\mathbb{R}$	real numbers (Definition 4)	decimal expansions

Set-builder notation

Exercise 2 (Natural numbers with small cubes)**.** List the natural numbers (Definition 3) n with $n^3 < 100$.

Solution

Solution 2. 1, 2, 3 and 4, whose cubes are 1, 8, 27 and 64. Every other natural number n has $n \geq 5$, so $n^3 \geq 125$, which is not less than 100.

Definition 5 (Set-builder notation)**.** For a set A and a statement $P(x)$ about an element x, **set-builder notation** (Wikipedia contributors 2026o) $\{x \in A : P(x)\}$ denotes the set of elements x of A for which $P(x)$ is true.

Example 3 (Even die rolls in set-builder notation)**.** $\{x \in \{1, 2, 3, 4, 5, 6\} : x \text{ is even}\} = \{2, 4, 6\}$ is the set of even rolls, and $\{x \in \mathbb{R} : x > 0\}$ is the set of positive real numbers.

Definition 6 (Empty set)**.** The **empty set** (Wikipedia contributors 2026f), denoted $\emptyset$, is the set that has no elements.

Remark 4 (Other notation for the empty set). Other sources write $\{\}$ for $\emptyset$ (Wikipedia contributors 2026f). Some sources call it the **null set**, but in measure theory a "null set" usually means a set of measure zero, which need not be empty. For example, the interval $[0, 0] = \{0\}$ has length $0 - 0 = 0$, so it is a null set in that sense, but it is not empty, because $0 \in \{0\}$.

Example 4 (Impossible die rolls)**.** No roll of a six-sided die is greater than 6, so $\{x \in \{1, 2, 3, 4, 5, 6\} : x > 6\} = \emptyset$.

Definition 7 (Set-builder notation with an expression)**.** For a set A and a rule $f(x)$ that gives an object for each element x of A, the notation $\{f(x) : x \in A\}$ stands for the set of the objects $f(x)$ for $x \in A$ (Wikipedia contributors 2026o). That is, an object y is an element of $\{f(x) : x \in A\}$ when $y = f(x)$ for some $x \in A$.

Example 5 (Two ways to build a set)**.**

- The numbers of Exercise 2 form the set $\{n \in \mathbb{N} : n^3 < 100\} = \{1, 2, 3, 4\}$ (Definition 5).
- Their cubes form the set $\{n^3 : n \in \{1, 2, 3, 4\}\} = \{1, 8, 27, 64\}$ (Definition 7).
- $\{2^k : k \in \mathbb{N}\} = \{2, 4, 8, 16, ...\}$, the powers of 2 after 2^0.
- $\{(-1)^k : k \in \mathbb{N}\} = \{-1, 1\}$: many values of k give the same value, and each value is one element of the set.

Cardinality

Exercise 3 (Letters in a word)**.** How many different letters does the word MISSISSIPPI use?

Solution

Solution 3. Four: M, I, S and P. The set of its letters is $\{\mathrm{M}, \mathrm{I}, \mathrm{S}, \mathrm{P}\}$, and listing a letter again, as the word does, does not add an element (Example 2).

Definition 8 (Finite set)**.** A set A is **finite** (Wikipedia contributors 2026g) if it is empty, or if, for some natural number n, its elements can be listed as a finite sequence (Definition: Finite sequence) $a_1, \ldots, a_n$ in which each element of A appears exactly once.

Definition 9 (Cardinality)**.** For a finite set A (Definition 8), the number n in such a list is the **cardinality** (Wikipedia contributors 2026b) of A, its number of elements, written $|A|$; the empty set has cardinality $|\emptyset| = 0$.

Remark 5 (Two meanings of $|\cdot|$)*.* Applied to a set, the bars $|A|$ mean the cardinality of A. Applied to a number, $|x|$ means its absolute value, or its modulus when x is complex. For example, $|\{-3\}| = 1$, but $|-3| = 3$.

Definition 10 (Infinite set)**.** A set that is not finite (Definition 8) is **infinite** (Wikipedia contributors 2026h).

Example 6 (Finite and infinite sets)**.**

- $\{2, 4, 6\}$ is finite, with $|\{2, 4, 6\}| = 3$.
- $\{1, 1, 2\} = \{1, 2\}$, so $|\{1, 1, 2\}| = 2$: a repeated listing does not count twice.
- $\emptyset$ is finite, with $|\emptyset| = 0$.
- The natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ are infinite: for any n, a list $a_1, \ldots, a_n$ of natural numbers misses the natural number $a_1 + \cdots + a_n + 1$, which is larger than every number in the list.

Remark 6 (Sets can be elements of sets). The elements of a set can themselves be sets. For example, $S = \{\{1, 2\}, 3, \{3\}\}$ has three elements: the set $\{1, 2\}$, the number 3, and the set $\{3\}$. So $\{1, 2\} \in S$, but $1 \notin S$, because 1 is an element of an element of S, not an element of S. And $3 \neq \{3\}$: one is a number, and the other is a set with one element. So $|S| = 3$ (Definition 9).

Definition 11 (Countable set)**.** A set is **countable** (Wikipedia contributors 2026e) if it is finite (Definition 8), or if its elements can be listed as a sequence (Definition: Sequence) $a_1, a_2, a_3, ...$ in which every element appears.

Definition 12 (Countably infinite set)**.** A set is **countably infinite** if it is countable and not finite.

Example 7 (Countable and uncountable sets)**.**

- $\{1, 2, 3, 4, 5, 6\}$ is countable, because it is finite.
- The non-negative integers $\{0, 1, 2, ...\}$ are countably infinite: the sequence $0, 1, 2, ...$ lists them.
- The integers are countably infinite too: the sequence $0, 1, -1, 2, -2, 3, -3, ...$ lists every one of them.

Remark 7 (Numbers with only one decimal expansion). These notes use one more fact about the real numbers without proving it. A real number has two decimal expansions only if one of them ends in repeating 9s and the other in repeating 0s, as $0.999...$ and $1.000...$ do (Remark 2). So a number whose decimal places are all 4s and 5s, such as $0.545454...$, has only one decimal expansion.

Theorem 1 (The real numbers are not countable)**.** *The set of real numbers $\mathbb{R}$ (Definition 4) is not countable (Definition 11) (Wikipedia contributors 2026a).*

Proof

Proof. Suppose, for contradiction (Definition: Proof by contradiction), that $\mathbb{R}$ is countable. $\mathbb{R}$ is not finite, because it contains the natural numbers, and a list of every real number would include a list of every natural number, which cannot be finite (Example 6). So the elements of $\mathbb{R}$ can be listed as a sequence $r_1, r_2, r_3, ...$ in which every real number appears. Write each r_n as a decimal expansion, choosing one if r_n has two. Let d be the number between 0 and 1 whose nth decimal place is

- 5, if the nth decimal place of r_n is not 5;
- 4, if the nth decimal place of r_n is 5.

Then d is a real number, so $d = r_m$ for some m. The decimal places of d are all 4s and 5s, so d has only one decimal expansion (Remark 7), and the expansion chosen for r_m must be that one. But the two expansions differ in the mth decimal place, by the choice of d. That is a contradiction, so $\mathbb{R}$ is not countable. □

Example 8 (The diagonal number for a list)**.** Suppose a list of real numbers starts

$$r_1 = 0.\mathbf{7}071\ldots$$
$$r_2 = 3.1\mathbf{4}15\ldots$$
$$r_3 = -2.71\mathbf{8}2\ldots$$
$$r_4 = 0.555\mathbf{5}\ldots$$

The bold digits are the nth decimal places of the r_n: 7, 4, 8 and 5. The proof of Theorem 1 builds $d = 0.5554\ldots$, which differs from r_1 in the first decimal place, from r_2 in the second, from r_3 in the third, and from r_4 in the fourth. However the list continues, d is not on it.

Subsets and power sets

Exercise 4 (An element that is also a subset)**.** Let $B = \{1, \{1\}\}$. A set A is a subset of B when every element of A is an element of B. Find a set that is both an element of B and a subset of B.

Solution

Solution 4. The set $\{1\}$ works. It is listed in B, so it is an element of B. And 1, the one element of $\{1\}$, is also listed in B, so every element of $\{1\}$ is an element of B.

Definition 13 (Subset)**.** A set A is a **subset** (Wikipedia contributors 2026p) of a set B, written $A \subseteq B$, if every element of A is an element of B.

Example 9 (Even die rolls)**.** The even die rolls form a subset of the possible rolls: $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\}$, because each of 2, 4, and 6 is a possible roll. The set $\{2, 4, 7\}$ is not a subset of $\{1, 2, 3, 4, 5, 6\}$, because 7 is not a possible roll.

Definition 14 (Strict subset, proper subset)**.** A set A is a **strict subset** (also called a **proper subset**) of a set B, written $A \subsetneq B$, if A is a subset of B and $A \neq B$.

Remark 8 (The symbol $\subset$). Sources disagree about the symbol $\subset$: some use it for "subset" ($\subseteq$), and others for "strict subset" ($\subsetneq$) (Wikipedia contributors 2026p). For example, $\{1, 2\} \subset \{1, 2\}$ is true in the first reading, because $\{1, 2\} \subseteq \{1, 2\}$, and false in the second, because the two sets are equal. These notes avoid $\subset$ and write $\subseteq$ or $\subsetneq$.

Example 10 (Strict subsets)**.**

- $\{2, 4, 6\} \subsetneq \{1, 2, 3, 4, 5, 6\}$, because $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\}$ (Example 9), and 1 is in the second set but not the first.
- $\mathbb{N} \subsetneq \mathbb{Z}$, because every natural number is an integer, and -1 is an integer that is not a natural number.
- $\mathbb{Z} \subseteq \mathbb{Z}$, because every integer is an integer, but $\mathbb{Z}$ is not a strict subset of itself.

Theorem 2 (The empty set is a subset of every set)**.** *For every set A, $\emptyset \subseteq A$.*

Proof

Proof. By Definition 13, $\emptyset \subseteq A$ fails only if some element of $\emptyset$ is not in A. The empty set has no elements, so no such element exists. □

Theorem 3 (Every set is a subset of itself)**.** *For every set A, $A \subseteq A$.*

Proof

Proof. Every element of A is an element of A, which is Definition 13 with $B = A$. □

Definition 15 (Power set)**.** The **power set** (Wikipedia contributors 2026l) of a set A, written $\mathcal{P}(A)$, is the set of all subsets of A:

$$\mathcal{P}(A) \stackrel{\text{def}}{=} \{S : S \subseteq A\}$$

Example 11 (The subsets of a three-element set)**.**

$$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Each element of $\mathcal{P}(\{a, b, c\})$ is a set. For example, $\{a\} \in \mathcal{P}(\{a, b, c\})$, but $a \notin \mathcal{P}(\{a, b, c\})$. The list includes $\emptyset$ (Theorem 2) and $\{a, b, c\}$ itself (Theorem 3).

Ordered pairs and Cartesian products

Exercise 5 (Sandwich orders)**.** A sandwich shop makes sandwiches on rye or wheat bread, with a filling of ham, cheese or tomato. List every choice of one bread and one filling, writing each choice as (bread, filling).

Solution

Solution 5. (rye, ham), (rye, cheese), (rye, tomato), (wheat, ham), (wheat, cheese) and (wheat, tomato).

Definition 16 (Ordered pair)**.** An **ordered pair** (Wikipedia contributors 2026k) (a, b) is a pair of objects listed in order, with a first and b second; two ordered pairs are equal, $(a, b) = (c, d)$, when $a = c$ and $b = d$.

Definition 17 (Tuple)**.** For a natural number n, an **n-tuple** (Wikipedia contributors 2026q) $(a_1, a_2, \dots, a_n)$ is a list of n objects in order. Two n-tuples are equal when their entries are equal in each position.

Example 12 (Tuples are not sets)**.**

- $(1, 2) \neq (2, 1)$, because their first entries differ, but $\{1, 2\} = \{2, 1\}$, because order does not matter in a set (Definition 1).
- $(4, 4, 1)$ is a 3-tuple whose first two entries are equal, but $\{4, 4, 1\} = \{1, 4\}$ has two elements.

Definition 18 (Cartesian product)**.** The **Cartesian product** (Wikipedia contributors 2026c) of sets A and B, written $A \times B$, is the set of ordered pairs whose first entry is in A and whose second entry is in B:

$$A \times B \stackrel{\text{def}}{=} \{(a, b) : a \in A \text{ and } b \in B\}$$

Definition 19 (Cartesian product of several sets)**.** For a natural number n and sets $A_1, A_2, \dots, A_n$, the **Cartesian product** (Wikipedia contributors 2026c) $A_1 \times A_2 \times \cdots \times A_n$ is the set of n-tuples whose ith entry is in A_i for each i:

$$A_1 \times \cdots \times A_n \stackrel{\text{def}}{=} \{(a_1, \dots, a_n) : a_i \in A_i \text{ for each } i\}$$

Example 13 (Cartesian products)**.**

- The choices in Exercise 5 are the Cartesian product $\{\text{rye}, \text{wheat}\} \times \{\text{ham}, \text{cheese}, \text{tomato}\}$ (Definition 18).
- $\{\text{ham}, \text{cheese}, \text{tomato}\} \times \{\text{rye}, \text{wheat}\}$ is a different set: it holds (ham, rye), which is not (rye, ham) (Example 12).
- $\mathbb{R} \times \mathbb{R}$ is the set of points (x, y) in the plane.
- Adding a choice of drink gives 3-tuples (Definition 19), such as $(\text{rye}, \text{ham}, \text{tea}) \in \{\text{rye}, \text{wheat}\} \times \{\text{ham}, \text{cheese}, \text{tomato}\} \times \{\text{tea}, \text{juice}\}$.

Set operations

Exercise 6 (Two sets from their overlap)**.** Find sets A and B whose common elements are exactly 2 and 3, and whose elements together are exactly 1, 2, 3 and 4.

Solution

Solution 6. $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ work: the elements in both are 2 and 3, and the elements in one or the other are 1, 2, 3 and 4. So do $A = \{1, 2, 3, 4\}$ and $B = \{2, 3\}$.

Definition 20 (Union)**.** The **union** (Wikipedia contributors 2026r) of sets A and B, written $A \cup B$, is the set of elements that are in A, in B, or in both:

$$A \cup B \stackrel{\text{def}}{=} \{x : x \in A \text{ or } x \in B\}$$

Definition 21 (Intersection)**.** The **intersection** (Wikipedia contributors 2026i) of sets A and B, written $A \cap B$, is the set of elements that are in both A and B:

$$A \cap B \stackrel{\text{def}}{=} \{x : x \in A \text{ and } x \in B\}$$

Definition 22 (Set difference)**.** The **set difference** (Wikipedia contributors 2026d) of sets A and B, written $A \setminus B$, is the set of elements of A that are not in B:

$$A \setminus B \stackrel{\text{def}}{=} \{x \in A : x \notin B\}$$

Remark 9 (Other notation for the set difference). Some sources write $A - B$ for $A \setminus B$ (Wikipedia contributors 2026d).

Example 14 (Combining sets of die rolls)**.** Let $A = \{2, 4, 6\}$ (the even rolls) and $B = \{1, 2, 3\}$ (the rolls of at most 3). Then:

- $A \cup B = \{1, 2, 3, 4, 6\}$, the rolls that are even or at most 3;
- $A \cap B = \{2\}$, the only roll that is both;
- $A \setminus B = \{4, 6\}$, the even rolls greater than 3;
- $\{2, 4, 6\} \cap \{1, 3, 5\} = \emptyset$: no roll is both even and odd.

Definition 23 (Complement)**.** Let A be a subset of a set S. The **complement** (Wikipedia contributors 2026d) of A in S is the set difference (Definition 22) $S \setminus A$: the elements of S that are not in A. When the set S is clear from context, the complement of A is written A^c:

$$A^c \stackrel{\text{def}}{=} S \setminus A = \{x \in S : x \notin A\}$$

Remark 10 (The universal set, and other notation for the complement). The set S in Definition 23 is sometimes called the *universal set*. Which universal set to use depends on the context: for a set of even integers it might be $\mathbb{Z}$, and for a set of letters it might be the 26 letters of the alphabet. Some sources write $\overline{A}$ for A^c (Wikipedia contributors 2026d).

Example 15 (Complements of sets of die rolls)**.** Let $S = \{1, 2, 3, 4, 5, 6\}$ be the possible rolls of a die.

- The complement of the even rolls $A = \{2, 4, 6\}$ is $A^c = \{1, 3, 5\}$, the odd rolls.
- The complement of S itself is $S^c = \emptyset$, and the complement of $\emptyset$ is $\emptyset^c = S$.

The complement depends on S: in $S = \{1, 2, 3, 4, 5, 6\}$, the complement of $\{2, 4, 6\}$ is $\{1, 3, 5\}$, but in $S = \{2, 4, 6, 8\}$, it is $\{8\}$.

Definition 24 (Venn diagram)**.** A **Venn diagram** draws sets as overlapping circles inside a rectangle, which stands for the set S that the sets are subsets of, as in Definition 23. Each region of the picture stands for the elements that are in some of the sets and not in the others.

Example 16 (Letters of two words)**.** Let L be the set of letters in the word LOGIC, and M the set of letters in the word MAGIC, inside the set S of the 26 letters of the alphabet. Figure 1 draws them.

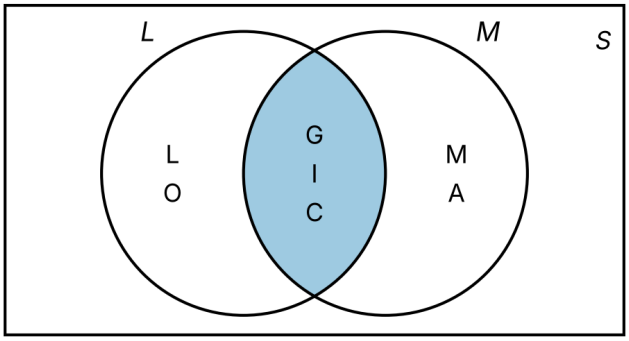


Figure 1: The letters of LOGIC (L) and of MAGIC (M), with $L \cap M$ shaded.

- The shaded region, inside both circles, is $L \cap M = \{\mathrm{G}, \mathrm{I}, \mathrm{C}\}$.
- The part of circle L outside circle M is $L \setminus M = \{\mathrm{L}, \mathrm{O}\}$.
- Everything inside either circle is $L \cup M = \{\mathrm{L}, \mathrm{O}, \mathrm{G}, \mathrm{I}, \mathrm{C}, \mathrm{M}, \mathrm{A}\}$.
- Everything in the rectangle outside circle L is L^c, the 21 letters not in LOGIC.

Counting subsets

Exercise 7 (Frozen yogurt toppings)**.** A frozen yogurt shop has five toppings: sprinkles, fudge, granola, berries and nuts. A customer picks two toppings and may pick the same topping twice. The order does not matter: sprinkles then fudge is the same choice as fudge then sprinkles. Counting such choices is a first step toward counting the subsets of a set.

> **Hint**
>
> Write each choice as a set in roster notation, such as $\{\text{sprinkles}, \text{fudge}\}$. Listing an element twice does not add an element (Example 2). An ordered pair of toppings keeps the order of the picks.

Compute the number of different choices.

Solution

Solution 7. There are 15 choices.
Each choice is a set of one or two toppings, so count the choices of two different toppings and the choices of one topping separately.

- **Two different toppings.** There are 5 choices for the first topping and 4 for the second, so there are $5 \cdot 4 = 20$ ordered pairs of different toppings. The sets $\{\text{sprinkles}, \text{fudge}\}$ and $\{\text{fudge}, \text{sprinkles}\}$ are equal, so each choice comes from exactly two ordered pairs. That leaves $20/2 = 10$ choices.
- **The same topping twice.** The choice is a set with one element, such as $\{\text{nuts}\}$, and there are 5 of these.

In all, there are $10 + 5 = 15$ choices.

Definition 25 (Binomial coefficient)**.** For a finite set A with n elements and a nonnegative integer k, the **binomial coefficient** $\binom{n}{k}$, read "n choose k", is the number of subsets of A that have exactly k elements:

$$\binom{n}{k} \stackrel{\text{def}}{=} |\{S : S \subseteq A \text{ and } |S| = k\}|$$

Example 17 (Subsets of $\{a, b, c\}$ by size)**.**
The set $\{a, b, c\}$ has 3 elements. Its subsets with exactly 2 elements are $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$, so $\binom{3}{2} = 3$. Its only subset with 0 elements is the empty set, so $\binom{3}{0} = 1$, and its only subset with 3 elements is $\{a, b, c\}$ itself, so $\binom{3}{3} = 1$.

Example 18 (Ordered pairs are not subsets)**.**
The set $\{a, b, c\}$ has 6 ordered pairs of different elements: (a, b), (b, a), (a, c), (c, a), (b, c) and (c, b). That count is not $\binom{3}{2}$, because (a, b) and (b, a) are different pairs but give the same subset $\{a, b\}$. The binomial coefficient $\binom{3}{2} = 3$ counts subsets, which ignore order (Example 17).

Theorem 4 (Choosing two elements)**.** *For every integer $n \geq 0$, the binomial coefficient $\binom{n}{2}$ is half of n times $n - 1$:*

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

Proof

Proof. Let A be a finite set with n elements. Count the ordered pairs (a, b) with $a \in A$, $b \in A$ and $a \neq b$. There are n choices for a, and for each of them there are $n - 1$ choices for b, so there are $n(n-1)$ such pairs.
The set $\{a, b\}$ is a subset of A with 2 elements, and it equals $\{b, a\}$. So each 2-element subset of A comes from exactly two of these ordered pairs, namely (a, b) and (b, a). Hence $n(n-1) = 2\binom{n}{2}$, and dividing both sides by 2 gives the claim. □

Example 19 (Pairs from three and from five elements)**.**
By Theorem 4, $\binom{3}{2} = \frac{3 \cdot 2}{2} = 3$, which matches the three subsets listed in Example 17. Likewise $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$, the number of ways to pick two different toppings from five in Solution 7.

Exercise 8 (Choosing two elements with repeats allowed)**.** Let S be a set with N elements. Choose two elements of S, where the same element may be chosen twice and the order does not matter. The five-topping choices of Exercise 7 are the case $N = 5$, and the same count arises whenever two items are drawn from N kinds with repeats allowed.

> **Hint**
>
> Split the choices by whether the two elements differ, as in Solution 7, and count the first kind with the binomial coefficient and Theorem 4.

Compute the number of different choices as a formula in N.

Solution

Solution 8. There are $\frac{N(N+1)}{2}$ choices, which is $\binom{N+1}{2}$.
As in Solution 7, each choice is a set $\{a, b\}$ with $a, b \in S$, and the set has one of two forms.

- **Two different elements.** These are the 2-element subsets of S, and there are $\binom{N}{2} = \frac{N(N-1)}{2}$ of them by Theorem 4.
- **The same element twice.** These are the sets $\{a\}$ with $a \in S$, and there are N of them.

Adding the two counts, one operation per line:

$$\begin{aligned}
\frac{N(N-1)}{2} + N &= \frac{N(N-1)}{2} + \frac{2N}{2} && \text{writing } N = \tfrac{2N}{2} \\
&= \frac{N(N-1) + 2N}{2} && \text{adding fractions with the same denominator} \\
&= \frac{N^2 - N + 2N}{2} && \text{distributing } N \\
&= \frac{N^2 + N}{2} && \text{combining } -N + 2N \\
&= \frac{N(N+1)}{2} && \text{factoring out } N \\
&= \binom{N+1}{2} && \text{taking } n = N+1 \text{ in the theorem}
\end{aligned}$$

The last line applies Theorem 4 with $n = N + 1$. For the five toppings, $N = 5$ gives $\frac{5 \cdot 6}{2} = 15$ choices, as in Solution 7.

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