# Predicates and Quantifiers

**For all, there exists, and the order they come in**

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**Source**

Adapted from Kameron Decker Harris's Fall 2026 lecture notes "Logic 2" and "Logic 3", which are typed transcriptions of Scott Wehrwein's Spring 2025 handwritten lecture notes. The convention that a conditional about a variable means "for all" comes from those notes; the exercises are new.
The definitions are on the proof writing page, which the other data science notes link to.

## Predicates

**Exercise 1** (Plug in a value)**.** Let $L(s)$ be the sentence "the string $s$ has more than 3 characters".

1. Is $L(s)$ true or false?
2. What are the truth values of $L(\texttt{"cat"})$, $L(\texttt{"logic"})$ and $L(\texttt{""})$?

Solution

*Solution* 1.

1. Neither: $L(s)$ has no truth value until we know which string $s$ is. It is a predicate (Definition: Predicate), not a proposition (Definition: Proposition, statement).
2. Each value of $s$ turns $L(s)$ into a proposition:
   - $L($`"cat"`$)$ is false, since `"cat"` has 3 characters.
   - $L($`"logic"`$)$ is true, since `"logic"` has 5 characters.
   - $L($`""`$)$ is false, since the empty string has 0 characters.

**Definition 1** (Domain of discourse, universe of discourse)**.** The **domain of discourse** (or **universe of discourse**) of a predicate $P(x)$ is the set of values its variable $x$ may take. A predicate in several variables has a domain of discourse for each variable.

*Remark* 1 (Predicates with more than one variable)*.* A predicate can have more than one variable: "$x < y$" is a predicate $P(x, y)$ in the two variables $x$ and $y$. Its domain is a set of pairs, such as the pairs $(x, y)$ of real numbers.

*Remark* 2 (Predicates in Racket)*.* A Racket procedure that returns `#t` or `#f` is also called a predicate, and by convention its name ends in `?`, as in `even?` and `string?`. The predicate $L(s)$ from Exercise 1 is

```
(define (long? s)
  (> (string-length s) 3))
```

and it is true of `"logic"`:

```
(long? "logic")
; #t
```

## Quantifiers

**Exercise 2** (All or some)**.** Let $S = \{2, 3, 5, 7\}$. Which of these four sentences are true?

1. Every element of $S$ is prime.
2. Every element of $S$ is odd.
3. Some element of $S$ is even.
4. Some element of $S$ is greater than 10.

Solution

*Solution* 2.

1. True: 2, 3, 5 and 7 are all prime.
2. False: 2 is not odd. One element that fails is enough.
3. True: 2 is even. One element that works is enough.
4. False: none of 2, 3, 5 and 7 is greater than 10.

**Definition 2** (Quantifier)**.** A **quantifier** (Wikipedia contributors 2026c) turns a predicate $P(x)$ (Definition: Predicate) about the elements $x$ of a set $S$ into a proposition (Definition: Proposition, statement). There are two, each defined next: the universal quantifier and the existential quantifier.

**Definition 3** (Universal quantifier)**.** The **universal quantifier** (Wikipedia contributors 2026d) "for all" is a quantifier (Definition 2): $\forall x \in S, P(x)$ ("for every $x$ in $S$, $P(x)$") is true when $P(x)$ is true for every element $x$ of $S$, and false otherwise.

**Definition 4** (Existential quantifier)**.** The **existential quantifier** (Wikipedia contributors 2026b) "there exists" is a quantifier (Definition 2): $\exists x \in S, P(x)$ ("there is an $x$ in $S$ such that $P(x)$") is true when $P(x)$ is true for at least one element $x$ of $S$, and false otherwise.

*Remark* 3 (Reading and writing quantifiers). English has many phrasings of each quantifier:

- $\forall$: "for all", "for every", "for each", "for any", "every", "all".
- $\exists$: "there exists", "there is", "for some", "some", "at least one".

Books also differ in punctuation: $\forall x \in S, P(x)$ is also written $\forall x \in S : P(x)$, $(\forall x \in S)\, P(x)$, or $\forall x \in S.\, P(x)$. When the domain is clear, it is often left out, as in $\forall x, P(x)$.

*Remark* 4 (Quantifiers in Racket). On a finite list, Racket computes the two quantifiers with `andmap` and `ormap`. Here they give the answers to parts 2 and 3 of Exercise 2:

```
(andmap odd? (list 2 3 5 7))
; #f
(ormap even? (list 2 3 5 7))
; #t
```

**Exercise 3** (Nothing to check)**.** Let $\emptyset$ be the empty set, the set with no elements. What are the truth values of $\forall x \in \emptyset, x > 5$ and $\exists x \in \emptyset, x > 5$?

**Hint**

What would make each one false?

Solution

*Solution* 3. $\forall x \in \emptyset, x > 5$ is true. It would be false only if some element of $\emptyset$ were not greater than 5, and $\emptyset$ has no elements.
$\exists x \in \emptyset, x > 5$ is false, since $\emptyset$ has no element at all, let alone one greater than 5.
Racket agrees: `(andmap positive? '())` returns `#t`, and `(ormap positive? '())` returns `#f`.

**Definition 5** (Vacuous truth)**.** A statement of the form "every element of $A$ has property $P$" is **vacuously true** (Wikipedia contributors 2026e) when $A$ is the empty set: $A$ has no elements, so no element of $A$ fails to have property $P$.

## Counterexamples

**Exercise 4** (A pattern that breaks)**.** Consider the claim "for every integer $n \geq 0$, $n^2 + n + 41$ is prime".

1. Check the claim for $n = 0, 1, 2, 3$.
2. Find an integer $n \geq 0$ for which the claim fails.

**Hint**

Look for an $n$ that makes every term of $n^2 + n + 41$ a multiple of 41.

Solution

*Solution* 4.

1. The values are 41, 43, 47 and 53, which are all prime.
2. Take $n = 41$: $41^2 + 41 + 41 = 41 \cdot (41 + 1 + 1) = 41 \cdot 43$, which is not prime. The value $n = 40$ also fails: $40^2 + 40 + 41 = 1681 = 41^2$.

The claim held for every value we checked in part 1, and it is still false: checking some cases cannot prove a universal claim.

**Definition 6** (Counterexample)**.** A **counterexample** (Wikipedia contributors 2026a) to a proposition of the form "for every $x$ in $A$, $P(x)$", where $P(x)$ is a predicate (Definition: Predicate), is an element $x$ of $A$ for which $P(x)$ is false. A single counterexample shows that the proposition is false.

*Remark* 5 (One example, or every element). Settling a quantified proposition takes one element in two cases, and every element of the domain in the other two:

- To show that $\exists x \in S, P(x)$ is true, it is enough to find one $x$ in $S$ with $P(x)$ true.
- To show that $\forall x \in S, P(x)$ is false, it is enough to find one $x$ in $S$ with $P(x)$ false.
- To show that $\forall x \in S, P(x)$ is true, show $P(x)$ for every $x$ in $S$.
- To show that $\exists x \in S, P(x)$ is false, show that $P(x)$ is false for every $x$ in $S$.

When $S$ is infinite, the last two need a proof rather than a check of each element.

## Quantified conditionals

**Exercise 5** (A hidden "for all")**.** Consider the sentence "if a string has more than 3 characters, then it is not empty". Let $L(s)$ be "$s$ has more than 3 characters" and $N(s)$ be "$s$ is not empty", for strings $s$.

1. Write the sentence in symbols, with its quantifier.
2. Is it true?
3. Is its converse (Definition: Converse) true?

Solution

*Solution* 5.

1. The sentence is about every string, so it is $\forall s, (L(s) \Rightarrow N(s))$.
2. True: a string with more than 3 characters has at least 1 character, so it is not empty.
3. The converse is $\forall s, (N(s) \Rightarrow L(s))$: "if a string is not empty, then it has more than 3 characters". It is false. The string `"ab"` is a counterexample: it is not empty, but it has only 2 characters.

*Remark* 6 (A conditional about a variable means "for all"). By convention, a conditional whose parts are predicates, such as "if $x > 3$, then $x > 1$", means the universal statement $\forall x, (x > 3 \Rightarrow x > 1)$, even though no quantifier is written. Mathematical statements leave out "for all" in this way all the time, so put it back before deciding whether the statement is true.

## Nested quantifiers

**Exercise 6** (Who takes what)**.** Three students take courses as follows.

| Student | Courses |
|---|---|
| Ana | 201, 241 |
| Ben | 241 |
| Cy | 201, 247 |

Let $E(s, c)$ be "student $s$ takes course $c$", where $s$ ranges over the students and $c$ over the courses 201, 241 and 247. Which of these are true?

1. $\forall s, \exists c, E(s, c)$
2. $\exists c, \forall s, E(s, c)$
3. $\forall c, \exists s, E(s, c)$
4. $\exists s, \forall c, E(s, c)$

Solution

*Solution* 6.

1. True: "every student takes some course". Ana takes 201, Ben takes 241, and Cy takes 201.
2. False: "some course is taken by every student". Ben does not take 201, Cy does not take 241, and Ana does not take 247.
3. True: "every course has some student". Ana takes 201 and 241, and Cy takes 247.
4. False: "some student takes every course". Ana does not take 247, Ben does not take 201, and Cy does not take 241.

Parts 1 and 2 use the same quantifiers and the same predicate in a different order, and they have different truth values.

*Remark* 7 (Order of quantifiers). Read nested quantifiers from left to right. In $\forall s, \exists c, E(s,c)$, the course $c$ is chosen after the student $s$, so it may depend on $s$. In $\exists c, \forall s, E(s,c)$, one course $c$ must work for every student.
Swapping two quantifiers of the same kind does not change the meaning: $\forall x, \forall y, P(x,y)$ means the same as $\forall y, \forall x, P(x,y)$, and likewise for $\exists$. Swapping a $\forall$ and an $\exists$ can change it, as parts 1 and 2 of Exercise 6 show.

**Exercise 7** (English to symbols)**.** Write each sentence in symbols, and say whether it is true.

1. Some integer $n$ satisfies $n^2 = 2n + 3$.
2. No integer is both positive and negative.
3. Every list of integers has a largest element.

Solution

*Solution* 7.

1. $\exists n \in \mathbb{Z}, n^2 = 2n + 3$. True: $3^2 = 9 = 2 \cdot 3 + 3$.
2. $\forall n \in \mathbb{Z}, \neg(n > 0 \wedge n < 0)$. True: an integer greater than 0 is not less than 0.
3. Let $\mathcal{L}$ be the set of lists of integers. The sentence says that each list $\ell$ has an element $m$ that is at least as large as every element $x$ of $\ell$:

$$\forall \ell \in \mathcal{L},\ \exists m \in \ell,\ \forall x \in \ell,\ x \leq m.$$

Here "$m \in \ell$" means "$m$ is an element of the list $\ell$". False: the empty list has no elements, so it has no largest element. The sentence is true for every nonempty list.

## References


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