# Induction and Recursion

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*Remark.* A proof by induction proves a proposition about every integer from some starting integer on. A sequence defined by a rule on its earlier terms is the usual place to use it. This page defines such sequences, then adds the stronger form, strong induction, that they often need.

## Recursively defined sequences

**Definition 1** (Recursively defined sequence)**.** Let $n_0$ be an integer and $m$ a positive integer. A sequence $a_{n_0}, a_{n_0+1}, \ldots$ is **recursively defined** (Wikipedia contributors 2026c) by two things:

1. **Initial terms:** specific values for $a_{n_0}, \ldots, a_{n_0+m-1}$.
2. **Recurrence:** a rule that gives $a_n$ from the $m$ entries $a_{n-1}, \ldots, a_{n-m}$ just before it, for every integer $n \geq n_0 + m$.

The initial terms and the recurrence together fix every term: the first $m$ entries are given, and each later entry follows from the $m$ entries before it.

**Example 1** (Powers of 3)**.** The sequence $a_0, a_1, \ldots$ with initial term $a_0 = 1$ and recurrence $a_n = 3a_{n-1}$ for $n \geq 1$ has $m = 1$, and its terms are $1, 3, 9, 27, \ldots$ A proof by induction shows $a_n = 3^n$ for every integer $n \geq 0$, using the recurrence in the inductive step: if $a_{n-1} = 3^{n-1}$, then $a_n = 3 \cdot 3^{n-1} = 3^n$.

**Definition 2** (Fibonacci sequence)**.** The **Fibonacci sequence** $F_1, F_2, \ldots$ (Wikipedia contributors 2026a) is the recursively defined sequence with $m = 2$, initial terms $F_1 = 1$ and $F_2 = 1$, and recurrence $F_n = F_{n-1} + F_{n-2}$ for every integer $n \geq 3$.

**Example 2** (The first Fibonacci numbers)**.** $F_3 = F_2 + F_1 = 1 + 1 = 2$, $F_4 = F_3 + F_2 = 2 + 1 = 3$, $F_5 = F_4 + F_3 = 3 + 2 = 5$, and $F_6 = F_5 + F_4 = 5 + 3 = 8$. So the sequence begins $1, 1, 2, 3, 5, 8, \ldots$

## Strong induction

**Definition 3** (Proof by strong induction)**.** Let $n_0$ be an integer, and let $P(n)$ be a predicate about the integers $n \geq n_0$. A **proof by strong induction** (Wikipedia contributors 2026b) on $n$ shows that $P(n)$ holds for every integer $n \geq n_0$ by showing one thing: for every integer $n \geq n_0$, if $P(k)$ holds for every integer $k$ with $n_0 \leq k < n$, then $P(n)$ holds. The assumption that $P(k)$ holds for every integer $k$ with $n_0 \leq k < n$ is the **strong induction hypothesis**.
For $n = n_0$ no integer $k$ satisfies $n_0 \leq k < n$, so the strong induction hypothesis is vacuously true, and the proof for $n = n_0$ is a base case. When the proof for $n$ needs particular earlier cases that do not exist for the first few $n$, those few cases are shown separately as further base cases.

**Example 3** (Every term of the Fibonacci sequence is less than a power of 2)**.** Let $F_1, F_2, \ldots$ be the Fibonacci sequence. To prove $F_n < 2^n$ for every $n \in \mathbb{N}$, let $P(n)$ be the inequality $F_n < 2^n$ and use Definition 3 with $n_0 = 1$.
**Base cases, $n = 1$ and $n = 2$.** The step below uses the two entries $F_{n-1}$ and $F_{n-2}$ before $F_n$, which do not exist for $n = 1$ and $n = 2$, so these two cases are shown directly:

$$F_1 = 1 < 2 = 2^1 \quad (1 < 2)$$
$$F_2 = 1 < 4 = 2^2. \quad (1 < 4)$$

**Step of the strong induction, $n \geq 3$.** Let $n \geq 3$, and suppose $P(k)$ holds for every integer $k$ with $1 \leq k < n$. Then $k = n - 1$ and $k = n - 2$ both satisfy $1 \leq k < n$, so

$F_{n-1} < 2^{n-1}$ and $F_{n-2} < 2^{n-2}$. Therefore

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} && \text{(definition of } F_n\text{, since } n \geq 3) \\ &< 2^{n-1} + 2^{n-2} && \text{(strong induction hypothesis, twice)} \\ &< 2^{n-1} + 2^{n-1} && (2^{n-2} < 2^{n-1}) \\ &= 2 \cdot 2^{n-1} && (a + a = 2a) \\ &= 2^n, && (2 \cdot 2^{n-1} = 2^n) \end{aligned}$$

which is $P(n)$. By Definition 3, $F_n < 2^n$ for every $n \in \mathbb{N}$.

## Further reading

- Velleman (2019) is a structured introduction to proof techniques, including proof by induction.

## References


Velleman, Daniel J. 2019. *How to Prove It: A Structured Approach*. 3rd ed. Cambridge University Press. https://doi.org/10.1017/9781108539890.

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