Proving Non-Conditional Statements

If and only if, existence, and disproof

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Source

Adapted from Kameron Decker Harris's Fall 2026 lecture notes "Proving Non-Conditional Statements, Disproof", which are typed transcriptions of Scott Wehrwein's Spring 2025 handwritten lecture notes. These come from those notes:

- the outline of an "if and only if" proof;
- the outline of an existence proof that names an element;
- proving that several propositions are equivalent with a cycle of conditionals;
- the constructive example of half of a positive number;
- the non-constructive example of an irrational power of an irrational number;
- disproving a proposition by proving its negation.

The exercises are new.

Proving if and only if

Exercise 1 (A sum of squares equal to zero)**.** Let x and y be real numbers. Prove: $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$ (Definition: If and only if).
Hint: prove the two directions one at a time.

Solution

Solution 1. **If $x = 0$ and $y = 0$, then $x^2 + y^2 = 0$.** Suppose $x = 0$ and $y = 0$. Then $x^2 + y^2 = 0^2 + 0^2 = 0$.
If $x^2 + y^2 = 0$, then $x = 0$ and $y = 0$. Suppose $x^2 + y^2 = 0$. A square of a real number is never negative, so $y^2 \geq 0$, and

$$\begin{aligned} x^2 &= -y^2 && \text{(subtract } y^2 \text{ from both sides of } x^2 + y^2 = 0) \\ &\leq 0. && (y^2 \geq 0) \end{aligned}$$

Also $x^2 \geq 0$, so $x^2 = 0$, and so $x = 0$. The same steps with x and y swapped give $y = 0$.

Theorem 1 (A biconditional is two conditionals)**.** *For propositions P and Q,*

$$(P \Leftrightarrow Q) \equiv ((P \Rightarrow Q) \wedge (Q \Rightarrow P)).$$

Proof

Proof. By the definitions of if and only if, implies and and, the truth table is:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

The last two columns match in every row, so the two propositions are logically equivalent. □

Example 1 (Proving both directions)**.** Claim: let n be an integer. Then n is even if and only if $n + 2$ is even.
By Theorem 1, it is enough to prove two conditionals.

- If n is even, then $n = 2k$ for some integer k (Definition: Even integer), so $n + 2 = 2k + 2 = 2(k + 1)$, and $k + 1$ is an integer (Remark: Facts about integers used without proof). So $n + 2$ is even.
- If $n + 2$ is even, then $n + 2 = 2m$ for some integer m, so $n = 2m - 2 = 2(m - 1)$, and $m - 1$ is an integer. So n is even.

Remark 1 (The shape of an "if and only if" proof). A proof of "P if and only if Q" has two parts:

> ($\Rightarrow$) Suppose P.
> ⋮
> Therefore Q.
>
> ($\Leftarrow$) Suppose Q.
> ⋮
> Therefore P.

Each part is a proof of a conditional, so each can be a direct proof (Definition: Direct proof), a proof by contrapositive (Definition: Proof by contrapositive), or a proof by contradiction (Definition: Proof by contradiction), and the two parts need not use the same method.

Theorem 2 (A cycle of conditionals)**.** *Let $P_1, P_2, \ldots, P_k$ be propositions. If $P_1 \Rightarrow P_2$, $P_2 \Rightarrow P_3$, ..., $P_{k-1} \Rightarrow P_k$ and $P_k \Rightarrow P_1$ are all true, then $P_1, \ldots, P_k$ all have the same truth value.*

Proof

Proof. There are two cases.

- None of $P_1, \ldots, P_k$ is true. Then all of them are false, so they all have the same truth value.
- Some P_i is true. By the definition of implies, for any P and Q, if $P \Rightarrow Q$ and P are both true, then Q is true. So $P_i \Rightarrow P_{i+1}$ makes P_{i+1} true, then $P_{i+1} \Rightarrow P_{i+2}$ makes P_{i+2} true, and so on around the cycle, with $P_k \Rightarrow P_1$ carrying the truth of P_k back to P_1. After at most $k-1$ steps, every proposition in the list is true.

□

Example 2 (Three ways to say "odd")**.** Let n be an integer, and consider

- P_1: n is odd;
- P_2: $n+1$ is even;
- P_3: $n-1$ is even.

By Theorem 2, three conditionals show that all three have the same truth value.

- $P_1 \Rightarrow P_2$: if $n = 2k+1$ (Definition: Odd integer), then $n+1 = 2k+2 = 2(k+1)$.
- $P_2 \Rightarrow P_3$: if $n+1 = 2m$, then $n-1 = 2m-2 = 2(m-1)$.
- $P_3 \Rightarrow P_1$: if $n-1 = 2j$, then $n = 2j+1$.

In each step the new multiplier, $k+1$, $m-1$ or j, is an integer (Remark: Facts about integers used without proof). Proving all six conditionals between pairs would have taken twice as many steps.

Proving existence

Exercise 2 (A solution in integers)**.** Show that there are integers x and y with $5x+3y = 1$.

Solution

Solution 2. Take $x = 2$ and $y = -3$. Then $5x + 3y = 5 \cdot 2 + 3 \cdot (-3) = 10 - 9 = 1$. So integers with $5x+3y = 1$ exist (Definition: Existential quantifier). Other pairs work too, such as $x = -1$ and $y = 2$.

Definition 1 (Constructive proof)**.** A **constructive proof** (Wikipedia contributors 2026) of a proposition $\exists x \in S,\ P(x)$ with an existential quantifier names a particular element of S and shows that P is true of it.

Remark 2 (Several existential quantifiers). A proposition can open with several existential quantifiers. A constructive proof of it names one element for each quantifier. For example, a constructive proof of "there are integers m and n with $m + n = 1$" names one value of m and one of n.

Example 3 (Half of a positive number)**.** Claim: for every positive real number x, there is a positive real number y with $y < x$.
Let x be a positive real number. We give a constructive proof of "there is a positive real number y with $y < x$" for this x: take $y = \frac{x}{2}$. Then $y > 0$, because $x > 0$, and

$$\begin{aligned} x - y &= x - \frac{x}{2} && (\text{substitute } y = \tfrac{x}{2}) \\ &= \frac{x}{2} && (\text{arithmetic}) \\ &> 0, && (x > 0) \end{aligned}$$

so $y < x$. The element named, $\frac{x}{2}$, depends on x, which a "for every x, there is a y" statement allows.
The solution to Exercise 2 is also a constructive proof: it names the integers 2 and -3.

Definition 2 (Non-constructive proof)**.** A **non-constructive proof** (Wikipedia contributors 2026) of a proposition $\exists x \in S,\ P(x)$ shows that it is true without naming any element x of S for which $P(x)$ is true. It is a proof that is not a constructive proof.

Example 4 (An irrational power of an irrational number)**.** Claim: there are irrational numbers x and y (Definition: Irrational numbers) for which x^y is rational (Definition: Rational numbers).

The number $\sqrt{2}$ is irrational (Theorem: $\sqrt{2}$ is irrational). Consider $\sqrt{2}^{\sqrt{2}}$, and take two cases (Definition: Proof by cases, proof by exhaustion).

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. Then $x = \sqrt{2}$ and $y = \sqrt{2}$ work.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. Then take $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, both irrational, and

$$\begin{aligned} x^y &= \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} && (\text{substitute } x \text{ and } y) \\ &= \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} && (\text{a power of a power multiplies the exponents}) \\ &= \sqrt{2}^{2} && (\sqrt{2}\cdot\sqrt{2} = 2) \\ &= 2, \end{aligned}$$

which is rational, since $2 = \frac{2}{1}$.

Every number is rational or irrational, so one of the cases holds, and the claim is true. The proof does not say which case holds, so it does not name a pair x, y that works.

Disproof

Exercise 3 (Prove or disprove)**.** Prove or disprove: for every real number x, if $x^2 > 1$, then $x > 1$.

Solution

Solution 3. The proposition is false. Take $x = -2$. Then $x^2 = 4 > 1$, but $x > 1$ is false. So $x = -2$ makes "if $x^2 > 1$, then $x > 1$" false (Definition: Implies), and so it is a counterexample (Definition: Counterexample) to the proposition.

Definition 3 (Disproof)**.** A **disproof** of a proposition P is a proof of its negation, $\neg P$.

Example 5 (No integer squares to two)**.** Claim to disprove: there is an integer n with $n^2 = 2$.
By Theorem: De Morgan's law for an existential quantifier, its negation is "for every integer n, $n^2 \neq 2$". We prove the negation by cases (Definition: Proof by cases, proof by exhaustion). Let n be an integer.

- If n is -1, 0 or 1, then n^2 is 1, 0 or 1, which is not 2.
- Otherwise $n \leq -2$ or $n \geq 2$, so $|n| \geq 2$, and $n^2 = |n|^2 \geq 2^2 = 4$, so $n^2 \neq 2$.

Exercise 4 (An equation with a solution)**.** Prove or disprove: there are integers m and n with $6m + 4n = 2$.

Solution

Solution 4. The proposition is true. Take $m = 1$ and $n = -1$. Then $6m+4n = 6-4 = 2$.

Exercise 5 (An equation without a solution)**.** Prove or disprove: there are integers m and n with $6m + 4n = 3$.

Solution

Solution 5. The proposition is false. Applying Theorem: De Morgan's law for an existential quantifier to each of its two quantifiers in turn, its negation is "for all integers m and n, $6m+4n \neq 3$", and we prove that. Let m and n be integers. Then $6m+4n = 2(3m+2n)$, and $3m+2n$ is an integer (Remark: Facts about integers used without proof), so $6m+4n$ is even (Definition: Even integer). But $3 = 2 \cdot 1 + 1$ is odd (Definition: Odd integer), and no integer is both even and odd (Remark: Facts about integers used without proof), so $6m + 4n \neq 3$.

References

Wikipedia contributors. 2026. *Constructive Proof — Wikipedia, the Free Encyclopedia.* https://en.wikipedia.org/wiki/Constructive_proof.