Induction and Recursion

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Remark. A proof by induction proves a proposition about every integer from some starting integer on. A sequence defined by a rule on its earlier terms is the usual place to use it. This page defines such sequences, then adds the stronger form, strong induction, that they often need.

Recursively defined sequences

Definition 1 (Recursively defined sequence)**.** Let n_0 be an integer and m a positive integer. A sequence $a_{n_0}, a_{n_0+1}, \ldots$ is **recursively defined** (Wikipedia contributors 2026c) by two things:

1. **Initial terms:** specific values for $a_{n_0}, \ldots, a_{n_0+m-1}$.
2. **Recurrence:** a rule that gives a_n from the m entries $a_{n-1}, \ldots, a_{n-m}$ just before it, for every integer $n \geq n_0 + m$.

The initial terms and the recurrence together fix every term: the first m entries are given, and each later entry follows from the m entries before it.

Example 1 (Powers of 3)**.** The sequence $a_0, a_1, \ldots$ with initial term $a_0 = 1$ and recurrence $a_n = 3a_{n-1}$ for $n \geq 1$ has $m = 1$, and its terms are $1, 3, 9, 27, \ldots$ A proof by induction shows $a_n = 3^n$ for every integer $n \geq 0$, using the recurrence in the inductive step: if $a_{n-1} = 3^{n-1}$, then $a_n = 3 \cdot 3^{n-1} = 3^n$.

Definition 2 (Fibonacci sequence)**.** The **Fibonacci sequence** $F_1, F_2, \ldots$ (Wikipedia contributors 2026a) is the recursively defined sequence with $m = 2$, initial terms $F_1 = 1$ and $F_2 = 1$, and recurrence $F_n = F_{n-1} + F_{n-2}$ for every integer $n \geq 3$.

Example 2 (The first Fibonacci numbers)**.** $F_3 = F_2 + F_1 = 1 + 1 = 2$, $F_4 = F_3 + F_2 = 2 + 1 = 3$, $F_5 = F_4 + F_3 = 3 + 2 = 5$, and $F_6 = F_5 + F_4 = 5 + 3 = 8$. So the sequence begins $1, 1, 2, 3, 5, 8, \ldots$

Strong induction

Definition 3 (Proof by strong induction)**.** Let n_0 be an integer, and let $P(n)$ be a predicate about the integers $n \geq n_0$. A **proof by strong induction** (Wikipedia contributors 2026b) on n shows that $P(n)$ holds for every integer $n \geq n_0$ by showing one thing: for every integer $n \geq n_0$, if $P(k)$ holds for every integer k with $n_0 \leq k < n$, then $P(n)$ holds.

The assumption that $P(k)$ holds for every integer k with $n_0 \leq k < n$ is the **strong induction hypothesis**.

For $n = n_0$ no integer k satisfies $n_0 \leq k < n$, so the strong induction hypothesis is vacuously true, and the proof for $n = n_0$ is a base case. When the proof for n needs particular earlier cases that do not exist for the first few n, those few cases are shown separately as further base cases.

Example 3 (Every term of the Fibonacci sequence is less than a power of 2)**.** Let $F_1, F_2, \ldots$ be the Fibonacci sequence. To prove $F_n < 2^n$ for every $n \in \mathbb{N}$, let $P(n)$ be the inequality $F_n < 2^n$ and use Definition 3 with $n_0 = 1$.

Base cases, $n = 1$ and $n = 2$. The step below uses the two entries F_{n-1} and F_{n-2} before F_n, which do not exist for $n = 1$ and $n = 2$, so these two cases are shown directly:

$$F_1 = 1 < 2 = 2^1 \quad (1 < 2)$$
$$F_2 = 1 < 4 = 2^2. \quad (1 < 4)$$

Step of the strong induction, $n \geq 3$. Let $n \geq 3$, and suppose $P(k)$ holds for every integer k with $1 \leq k < n$. Then $k = n - 1$ and $k = n - 2$ both satisfy $1 \leq k < n$, so $F_{n-1} < 2^{n-1}$ and $F_{n-2} < 2^{n-2}$. Therefore

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} && \text{(definition of } F_n\text{, since } n \geq 3) \\ &< 2^{n-1} + 2^{n-2} && \text{(strong induction hypothesis, twice)} \\ &< 2^{n-1} + 2^{n-1} && (2^{n-2} < 2^{n-1}) \\ &= 2 \cdot 2^{n-1} && (a + a = 2a) \\ &= 2^n, && (2 \cdot 2^{n-1} = 2^n) \end{aligned}$$

which is $P(n)$. By Definition 3, $F_n < 2^n$ for every $n \in \mathbb{N}$.

Further reading

- Velleman (2019) is a structured introduction to proof techniques, including proof by induction.

References

Velleman, Daniel J. 2019. *How to Prove It: A Structured Approach*. 3rd ed. Cambridge University Press. https://doi.org/10.1017/9781108539890.

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