Proofs with Sets

Membership, subsets and set equality

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Source

Adapted from Kameron Decker Harris's Fall 2026 lecture notes "Proofs Involving Sets", which are typed transcriptions of Scott Wehrwein's Spring 2025 handwritten lecture notes. These come from those notes:

- what to show to prove that an element is in a set given in set-builder notation;
- the direct and contrapositive outlines of a subset proof;
- proving set equality by proving that each set is a subset of the other;
- the theorem that a nonempty factor cancels from both sides of an equation between Cartesian products, with the proof of its first half.

Everything else is new: the exercises, the other examples and theorems, the second half of that proof, the membership test for a set built with an expression, and the section on disproving a claim about sets.

Proving membership

Exercise 1 (In or out)**.** Let $T = \{3k + 2 : k \in \mathbb{Z}\}$ (Definition: Set-builder notation with an expression). Prove that $17 \in T$ and that $19 \notin T$.

Solution

Solution 1. $17 \in T$**.** An element of T is a number of the form $3k + 2$ with k an integer. Take $k = 5$, which is an integer. Then $3k + 2 = 3 \cdot 5 + 2 = 17$, so $17 \in T$.
$19 \notin T$**.** Suppose, for contradiction, that $19 \in T$ (Definition: Proof by contradiction). Then $19 = 3k + 2$ for some integer k. Subtracting 2 from both sides gives $3k = 17$, so

$k = 17/3$. But $17/3$ is not an integer, since $3 \cdot 5 = 15 < 17 < 18 = 3 \cdot 6$. This contradicts that k is an integer, so $19 \notin T$.

Remark 1 (What to show for membership). To prove that a is an element of a set, show that a passes the set's test:

- for $S = \{x \in A : P(x)\}$ (Definition: Set-builder notation), show that $a \in A$ and that $P(a)$ is true;
- for $S = \{x : P(x)\}$, the same notation with no set A named, show that $P(a)$ is true;
- for $S = \{f(k) : k \in A\}$ (Definition: Set-builder notation with an expression), find some $k \in A$ with $f(k) = a$.

The last case is an existence proof, so it usually names the k (Definition: Constructive proof). To prove that $a \notin S$, prove the negation of the test, often by contradiction, as in Solution 1.

Example 1 (A root of a polynomial)**.** Let $S = \{x \in \mathbb{R} : 2x^2 - x = 0\}$. Then $\frac{1}{2} \in S$. The set has the form $\{x \in A : P(x)\}$, so there are two things to show (Remark 1). First, $\frac{1}{2}$ is a real number. Second,

$$\begin{aligned} 2\left(\frac{1}{2}\right)^2 - \frac{1}{2} &= 2 \cdot \frac{1}{4} - \frac{1}{2} \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0. \end{aligned}$$

Proving a subset

Exercise 2 (A disk inside a square)**.** Let $D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq 1\}$, the points of the plane at distance at most 1 from the origin. Prove that $D \subseteq [-1, 1] \times [-1, 1]$, where $[-1, 1] = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ (Definition: Subset, Definition: Cartesian product, Definition: Closed interval).

Solution

Solution 2. Let $(a, b) \in D$, so a and b are real numbers with $a^2 + b^2 \leq 1$. Then

$$\begin{aligned} a^2 &\leq a^2 + b^2 && (b^2 \geq 0) \\ &\leq 1. && ((a,b) \in D) \end{aligned}$$

If $a > 1$, then $a^2 = a \cdot a > 1 \cdot 1 = 1$, and if $a < -1$, then $a^2 = (-a)^2 > 1$. Neither is possible, so $-1 \leq a \leq 1$, that is, $a \in [-1, 1]$. The same steps with a and b swapped give $b \in [-1, 1]$. So $(a, b) \in [-1, 1] \times [-1, 1]$. Every element of D is in $[-1, 1] \times [-1, 1]$, so $D \subseteq [-1, 1] \times [-1, 1]$.

Remark 2 (The shape of a subset proof). By Definition: Subset, $A \subseteq B$ means the conditional "if $a \in A$, then $a \in B$" is true for every a. So a proof of $A \subseteq B$ is a proof of that conditional, either directly (Definition: Direct proof):

> Suppose $a \in A$.
> ⋮
> Therefore $a \in B$.
> So $A \subseteq B$.

or by contrapositive (Definition: Proof by contrapositive):

> Suppose $a \notin B$.
> ⋮
> Therefore $a \notin A$.
> So $A \subseteq B$.

The element a is arbitrary: the proof may use only that $a \in A$ (or that $a \notin B$), nothing else about it.

Example 2 (A subset proof by contrapositive)**.** Claim: $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y > 2\} \subseteq \{(x, y) \in \mathbb{R} \times \mathbb{R} : x > 1 \text{ or } y > 1\}$. If two numbers add up to more than 2, at least one of them is more than 1.
We use the contrapositive outline of Remark 2. Suppose $p \notin \{(x, y) \in \mathbb{R} \times \mathbb{R} : x > 1 \text{ or } y > 1\}$. If p is not a pair of real numbers, then p is not in the left-hand set either, since every element of that set is a pair of real numbers. Otherwise $p = (a, b)$ with a and b real, and "$a > 1$ or $b > 1$" is false. By De Morgan's laws (Theorem: De Morgan's laws for propositions), $a > 1$ is false and $b > 1$ is false, that is, $a \leq 1$ and $b \leq 1$, so $a + b \leq 1 + 1 = 2$. So $p \notin \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y > 2\}$. So the left-hand set is a subset of the right-hand set.
The contrapositive turns the conclusion "$x > 1$ or $y > 1$", which does not say which number is large, into the hypothesis "$a \leq 1$ and $b \leq 1$", which says something about both.

Theorem 1 (Subsets of subsets)**.** *Let A, B and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof

Proof. Suppose $A \subseteq B$ and $B \subseteq C$, and let $a \in A$. Since $A \subseteq B$, $a \in B$ (Definition: Subset). Since $B \subseteq C$, $a \in C$. So every element of A is in C, and $A \subseteq C$. □

Example 3 (A chain of subsets)**.** Let D be the disk of Exercise 2, and let $H = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y \leq 2\}$. Solution 2 shows that $D \subseteq [-1, 1] \times [-1, 1]$. Every point (x, y) of $[-1, 1] \times [-1, 1]$ has $x \leq 1$ and $y \leq 1$, so $x + y \leq 2$, and so $[-1, 1] \times [-1, 1] \subseteq H$. By Theorem 1, $D \subseteq H$.

Proving set equality

Exercise 3 (Absorbing an intersection)**.** Let A and B be sets. Prove that $A \cup (A \cap B) = A$ (Definition: Union, Definition: Intersection).
Hint: prove that each side is a subset of the other.

Solution

Solution 3. $A \cup (A \cap B) \subseteq A$**.** Let $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$. In the first case, $x \in A$. In the second case, $x \in A$ and $x \in B$, so again $x \in A$. Either way, $x \in A$ (Definition: Proof by cases, proof by exhaustion).
$A \subseteq A \cup (A \cap B)$**.** Let $x \in A$. Then "$x \in A$ or $x \in A \cap B$" is true, since its first part is true (Definition: Or), so $x \in A \cup (A \cap B)$.
Each side is a subset of the other, so an element is in one side exactly when it is in the other, and the two sets are equal (Definition: Set).

Theorem 2 (Equal sets are subsets of each other)**.** *Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Proof

Proof. By Definition: Set, $A = B$ means that every x has $x \in A \Leftrightarrow x \in B$. By Theorem: A biconditional is two conditionals, that biconditional is the same as "$x \in A \Rightarrow x \in B$, and $x \in B \Rightarrow x \in A$". A conjunction is true for every x exactly when each of its two parts

is true for every x. The first conditional, for every x, is $A \subseteq B$ (Definition: Subset), and the second, for every x, is $B \subseteq A$. □

Remark 3 (The shape of a set equality proof). By Theorem 2, a proof of $A = B$ has two parts, each a subset proof (Remark 2):

> ($\subseteq$) Suppose $a \in A$.
> ⋮
> Therefore $a \in B$.
>
> ($\supseteq$) Suppose $a \in B$.
> ⋮
> Therefore $a \in A$.
>
> So $A = B$.

This is the "if and only if" outline of Remark: The shape of an "if and only if" proof, applied to $a \in A$ and $a \in B$.

Example 4 (Two kinds of odd number)**.** Claim: $\{4k+1 : k \in \mathbb{Z}\} \cup \{4k+3 : k \in \mathbb{Z}\} = \{2m+1 : m \in \mathbb{Z}\}$.
($\subseteq$) Let x be in the left-hand side. If $x = 4k+1$ for some integer k, then $x = 2(2k)+1$. If $x = 4k+3$ for some integer k, then $x = 2(2k+1)+1$. Both $2k$ and $2k+1$ are integers (Remark: Facts about integers used without proof), so either way x is in the right-hand side.
($\supseteq$) Let $x = 2m+1$ for some integer m. The integer m is even or odd (Remark: Facts about integers used without proof).

- If m is even, then $m = 2j$ for some integer j (Definition: Even integer), so $x = 2(2j)+1 = 4j+1$, and $x \in \{4k+1 : k \in \mathbb{Z}\}$.
- If m is odd, then $m = 2j+1$ for some integer j (Definition: Odd integer), so $x = 2(2j+1)+1 = 4j+3$, and $x \in \{4k+3 : k \in \mathbb{Z}\}$.

Either way, x is in the left-hand side. So the two sets are equal (Theorem 2).

Theorem 3 (A subset is its own intersection)**.** *Let A and B be sets. Then $A \subseteq B$ if and only if $A \cap B = A$.*

Proof

Proof. ($\Rightarrow$) Suppose $A \subseteq B$. We show $A \cap B = A$ with Theorem 2. If $x \in A \cap B$, then $x \in A$ (Definition: Intersection). If $x \in A$, then $x \in B$, since $A \subseteq B$, so $x \in A \cap B$.
($\Leftarrow$) Suppose $A \cap B = A$, and let $x \in A$. Then $x \in A \cap B$, since the two sets are equal, so $x \in B$. So $A \subseteq B$. □

Example 5 (Intersecting with a larger set)**.** Let $B = \{1, 2, 3, 4\}$.

- $\{2, 4\} \subseteq B$, and $\{2, 4\} \cap B = \{2, 4\}$, as Theorem 3 says.
- $\{2, 5\} \not\subseteq B$, since $5 \notin B$, and $\{2, 5\} \cap B = \{2\}$, which is not $\{2, 5\}$.

Theorem 4 (Cancelling a factor of a Cartesian product)**.** *Let A, B and C be sets with $C \neq \emptyset$. If $A \times C = B \times C$, then $A = B$ (Definition: Cartesian product).*

Proof

Proof. Suppose $A \times C = B \times C$. Since $C \neq \emptyset$, there is some $c \in C$ (Definition: Empty set).
($\subseteq$) Let $a \in A$. Then $(a, c) \in A \times C$. Since $A \times C = B \times C$, also $(a, c) \in B \times C$, so $(a, c) = (b, c')$ for some $b \in B$ and $c' \in C$ (Definition: Cartesian product). Two ordered pairs are equal only when their first entries are equal (Definition: Ordered pair), so $a = b$, and $a \in B$.
($\supseteq$) Let $b \in B$. Then $(b, c) \in B \times C = A \times C$, so $b \in A$, by the same steps with A and B swapped.
So $A = B$ (Theorem 2). □

Example 6 (Why C must not be empty)**.** The condition $C \neq \emptyset$ in Theorem 4 is needed. Let $A = \{1\}$, $B = \{2\}$ and $C = \emptyset$. A pair in $A \times C$ would need a second entry in C, and C has none, so $A \times C = \emptyset$, and likewise $B \times C = \emptyset$. So $A \times C = B \times C$, but $A \neq B$.
The proof uses the condition in its first step, to find an element $c \in C$.

Disproving a claim about sets

Exercise 4 (Regrouping differences)**.** Show that the claim "for all sets A, B and C, $A - (B - C) = (A - B) - C$" is false (Definition: Set difference).

Solution

Solution 4. A counterexample (Definition: Counterexample) is one choice of A, B and C for which the two sides differ. Let $A = B = C = \{1\}$. Then

$$\begin{aligned} A - (B - C) &= \{1\} - (\{1\} - \{1\}) \\ &= \{1\} - \emptyset \\ &= \{1\}, \end{aligned}$$

while

$$\begin{aligned} (A - B) - C &= (\{1\} - \{1\}) - \{1\} \\ &= \emptyset - \{1\} \\ &= \emptyset. \end{aligned}$$

The two sides differ, so the claim is false.

Remark 4 (Disproving a claim about all sets). A claim of the form "for all sets A, B, …, some equation holds" is disproved by one choice of sets for which it fails (Definition: Disproof). Small sets, such as $\emptyset$ and $\{1\}$, are a good place to look. A Venn diagram (Definition: Venn diagram) can suggest where to look: shade each side, find a region shaded on one side only, and put an element in that region.

Example 7 (A union does not cancel)**.** Claim to disprove: for all sets A, B and C, if $A \cup B = A \cup C$, then $B = C$.
Let $A = \{1\}$, $B = \{1\}$ and $C = \emptyset$. Then $A \cup B = \{1\}$ and $A \cup C = \{1\}$, so $A \cup B = A \cup C$, but $B \neq C$. So the claim is false.

References