Relations

Last modified: 2026-11-06 16:28:53 (PST)

Remark. A relation says which pairs of elements of a set are related. This page defines the properties that make a relation behave like equality, and the groups of elements that such a relation puts together.

Properties of relations

Definition 1 (Reflexive relation)**.** Let $\sim$ be a relation on a set S. The relation $\sim$ is **reflexive** (Wikipedia contributors 2026c) if $a \sim a$ for every $a \in S$.

Example 1 (Reflexive and non-reflexive relations on the real numbers)**.** On $\mathbb{R}$, the relation $=$ is reflexive, since $a = a$ for every real number a. The relation $\leq$ is also reflexive, since $a \leq a$ for every real number a. The relation $<$ is not reflexive, since $2 < 2$ is false.

Definition 2 (Symmetric relation)**.** Let $\sim$ be a relation on a set S. The relation $\sim$ is **symmetric** (Wikipedia contributors 2026d) if, for all $a, b \in S$, $a \sim b$ implies $b \sim a$.

Example 2 (Symmetric and non-symmetric relations on the real numbers)**.** On $\mathbb{R}$, the relation $=$ is symmetric: whenever $a = b$, also $b = a$. The relation $\leq$ is not symmetric, since $2 \leq 5$ is true but $5 \leq 2$ is false.

Equivalence relations

Definition 3 (Equivalence relation)**.** Let $\sim$ be a relation on a set S. The relation $\sim$ is an **equivalence relation** (Wikipedia contributors 2026b) if it is reflexive, symmetric, and transitive.

Example 3 (Integers with an even difference)**.** On the integers, define the relation $\sim$ by $a \sim b$ when $a - b$ is even. For example, $7 \sim 3$, since $7 - 3 = 4 = 2 \cdot 2$, but $7 \not\sim 4$, since $7 - 4 = 3$ is odd. The relation $\sim$ is an equivalence relation.
Reflexive. For every integer a, $a - a = 0 = 2 \cdot 0$, so $a \sim a$.
Symmetric. Suppose $a \sim b$, so $a - b = 2k$ for some integer k. Then

$$\begin{aligned} b - a &= -(a - b) && \text{(negate the difference)} \\ &= -2k && (a - b = 2k) \\ &= 2(-k), && \text{(factor out 2)} \end{aligned}$$

and $-k$ is an integer, so $b \sim a$.
Transitive. Suppose $a \sim b$ and $b \sim c$, so $a - b = 2j$ and $b - c = 2k$ for some integers j and k. Then

$$\begin{aligned} a - c &= (a - b) + (b - c) && \text{(the } b \text{ terms cancel)} \\ &= 2j + 2k && (a - b = 2j \text{ and } b - c = 2k) \\ &= 2(j + k), && \text{(distributive law)} \end{aligned}$$

and $j + k$ is an integer, so $a \sim c$.

Definition 4 (Equivalence class)**.** Let $\sim$ be an equivalence relation on a set S, and let $a \in S$. The **equivalence class** (Wikipedia contributors 2026a) of a is the set of elements of S related to a:

$$[a] \stackrel{\text{def}}{=} \{b \in S : b \sim a\}.$$

Example 4 (Equivalence classes of integers with an even difference)**.** Let $\sim$ be the equivalence relation of Example 3 on the integers. The equivalence class $[0]$ holds the integers b with $b - 0$ even, which are the even integers: $[0] = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. The class $[1]$ holds the integers b with $b - 1$ even, which are the odd integers: $[1] = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$. The class $[2]$ is the same set as $[0]$, since $2 \sim 0$.

References

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