

Proving Non-Conditional Statements

If and only if, existence, and disproof

Last modified: 2026-10-27 13:24:37 (PDT)

i Source

Adapted from Kameron Decker Harris's Fall 2026 lecture notes "Proving Non-Conditional Statements, Disproof", which are typed transcriptions of Scott Wehrwein's Spring 2025 handwritten lecture notes. These come from those notes:

- the outline of an "if and only if" proof;
- the outline of an existence proof that names an element;
- proving that several propositions are equivalent with a cycle of conditionals;
- the constructive example of half of a positive number;
- the non-constructive example of an irrational power of an irrational number;
- disproving a proposition by proving its negation.

The exercises are new.

Proving if and only if

Exercise 1 (A sum of squares equal to zero). Let x and y be real numbers. Prove: $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$ ([Definition: If and only if](#)).

Hint: prove the two directions one at a time.

Solution

Solution 1. **If $x = 0$ and $y = 0$, then $x^2 + y^2 = 0$.** Suppose $x = 0$ and $y = 0$. Then $x^2 + y^2 = 0^2 + 0^2 = 0$.

If $x^2 + y^2 = 0$, then $x = 0$ and $y = 0$. Suppose $x^2 + y^2 = 0$. A square of a real number is never negative, so $y^2 \geq 0$, and

$$\begin{aligned} x^2 &= -y^2 && \text{(subtract } y^2 \text{ from both sides of } x^2 + y^2 = 0) \\ &\leq 0. && (y^2 \geq 0) \end{aligned}$$

Also $x^2 \geq 0$, so $x^2 = 0$, and so $x = 0$. The same steps with x and y swapped give $y = 0$.

Theorem 1 (A biconditional is two conditionals). *For propositions P and Q ,*

$$(P \Leftrightarrow Q) \equiv ((P \Rightarrow Q) \wedge (Q \Rightarrow P)).$$

i Proof

Proof. The fifth and last columns of this truth table ([Definition: Truth table](#)) match in every row ([Definition: If and only if](#), [Definition: Implies](#), [Definition: And](#)):

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge$ $(Q \Rightarrow P)$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

So the two propositions are logically equivalent ([Definition: Logical equivalence](#)). □

Example 1 (Proving both directions). Claim: let n be an integer. Then n is even if and only if $n + 2$ is even.

By Theorem 1, it is enough to prove two conditionals.

- If n is even, then $n = 2k$ for some integer k ([Definition: Even integer](#)), so $n + 2 = 2k + 2 = 2(k + 1)$, and $k + 1$ is an integer ([Remark: Facts about integers used without proof](#)). So $n + 2$ is even.
- If $n + 2$ is even, then $n + 2 = 2m$ for some integer m , so $n = 2m - 2 = 2(m - 1)$, and $m - 1$ is an integer. So n is even.

Remark 1 (The shape of an “if and only if” proof). A proof of “ P if and only if Q ” has two parts:

($\Rightarrow$) Suppose P .
 $\vdots$
 Therefore Q .

($\Leftarrow$) Suppose Q .

$\vdots$

Therefore P .

Each part is a proof of a conditional, so each can be a direct proof ([Definition: Direct proof](#)), a proof by contrapositive ([Definition: Proof by contrapositive](#)), or a proof by contradiction ([Definition: Proof by contradiction](#)), and the two parts need not use the same method.

Theorem 2 (A cycle of conditionals). *Let $P_1, P_2, \dots, P_k$ be propositions. If $P_1 \Rightarrow P_2$, $P_2 \Rightarrow P_3$, ..., $P_{k-1} \Rightarrow P_k$ and $P_k \Rightarrow P_1$ are all true, then $P_1, \dots, P_k$ all have the same truth value.*

Proof

Proof. Suppose one of the propositions, P_i , is true. By modus ponens ([Theorem: Common rules of inference](#)), $P_i \Rightarrow P_{i+1}$ makes P_{i+1} true, then $P_{i+1} \Rightarrow P_{i+2}$ makes P_{i+2} true, and so on around the cycle, with $P_k \Rightarrow P_1$ leading from P_k back to P_1 , until every proposition in the list is true. So either all of $P_1, \dots, P_k$ are true, or none of them is true, and then all of them are false. $\square$

Example 2 (Three ways to say “odd”). Let n be an integer, and consider

- P_1 : n is odd;
- P_2 : $n + 1$ is even;
- P_3 : $n - 1$ is even.

By Theorem 2, three conditionals show that all three have the same truth value.

- $P_1 \Rightarrow P_2$: if $n = 2k + 1$ ([Definition: Odd integer](#)), then $n + 1 = 2k + 2 = 2(k + 1)$.
- $P_2 \Rightarrow P_3$: if $n + 1 = 2m$, then $n - 1 = 2m - 2 = 2(m - 1)$.
- $P_3 \Rightarrow P_1$: if $n - 1 = 2j$, then $n = 2j + 1$.

In each step the new multiplier, $k + 1$, $m - 1$ or j , is an integer ([Remark: Facts about integers used without proof](#)). Proving all six conditionals between pairs would have taken twice as many steps.

Proving existence

Exercise 2 (A solution in integers). Show that there are integers x and y with $5x + 3y = 1$.

Solution

Solution 2. Take $x = 2$ and $y = -3$. Then $5x + 3y = 5 \cdot 2 + 3 \cdot (-3) = 10 - 9 = 1$. So integers with $5x + 3y = 1$ exist ([Definition: Existential quantifier](#)). Other pairs work too, such as $x = -1$ and $y = 2$.

Definition 1 (Constructive proof). A **constructive proof** of a proposition $\exists x \in S, P(x)$ ([Definition: Existential quantifier](#)) names a particular element of S and shows that P is true of it (Wikipedia contributors 2026). A proposition with several existential quantifiers, such as “there are integers m and n with $m + n = 1$ ”, gets a constructive proof that names one element for each quantifier.

Example 3 (Half of a positive number). Claim: for every positive real number x , there is a positive real number y with $y < x$.

Let x be a positive real number. We give a constructive proof of “there is a positive real number y with $y < x$ ” for this x : take $y = \frac{x}{2}$. Then $y > 0$, because $x > 0$, and

$$\begin{aligned} x - y &= x - \frac{x}{2} && \text{(substitute } y = \frac{x}{2} \text{)} \\ &= \frac{x}{2} && \text{(arithmetic)} \\ &> 0, && (x > 0) \end{aligned}$$

so $y < x$. The element named, $\frac{x}{2}$, depends on x , which a “for every x , there is a y ” statement allows.

The solution to Exercise 2 is also a constructive proof: it names the integers 2 and -3 .

Definition 2 (Non-constructive proof). A **non-constructive proof** of a proposition $\exists x \in S, P(x)$ shows that it is true without naming any element x of S for which $P(x)$ is true (Wikipedia contributors 2026).

Example 4 (An irrational power of an irrational number). Claim: there are irrational numbers x and y ([Definition: Irrational numbers](#)) for which x^y is rational ([Definition: Rational numbers](#)).

The number $\sqrt{2}$ is irrational ([Theorem: \$\sqrt{2}\$ is irrational](#)). Consider $\sqrt{2}^{\sqrt{2}}$, and take two

cases ([Definition: Proof by cases](#)).

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. Then $x = \sqrt{2}$ and $y = \sqrt{2}$ work.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. Then take $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, both irrational, and

$$\begin{aligned}x^y &= \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} && \text{(substitute } x \text{ and } y\text{)} \\&= \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} && \text{(a power of a power multiplies the exponents)} \\&= \sqrt{2}^2 && (\sqrt{2} \cdot \sqrt{2} = 2) \\&= 2,\end{aligned}$$

which is rational, since $2 = \frac{2}{1}$.

Every number is rational or irrational, so one of the cases holds, and the claim is true.

The proof does not say which case holds, so it does not name a pair x, y that works.

Disproof

Exercise 3 (Prove or disprove). Prove or disprove: for every real number x , if $x^2 > 1$, then $x > 1$.

Solution

Solution 3. The proposition is false. Take $x = -2$. Then $x^2 = 4 > 1$, but $x > 1$ is false. So $x = -2$ makes “if $x^2 > 1$, then $x > 1$ ” false ([Definition: Implies](#)), and so it is a counterexample ([Definition: Counterexample](#)) to the proposition.

Definition 3 (Disproof). A **disproof** of a proposition P is a proof ([Definition: Proof](#)) of its negation, $\neg P$ ([Definition: Not](#)).

Example 5 (No integer squares to two). Claim to disprove: there is an integer n with $n^2 = 2$.

By [Theorem: Negating a quantifier](#), its negation is “for every integer n , $n^2 \neq 2$ ”. We prove the negation by cases ([Definition: Proof by cases](#)). Let n be an integer.

- If n is -1 , 0 or 1 , then n^2 is 1 , 0 or 1 , which is not 2 .
- Otherwise $n \leq -2$ or $n \geq 2$, so $|n| \geq 2$, and $n^2 = |n|^2 \geq 2^2 = 4$, so $n^2 \neq 2$.

Exercise 4 (An equation with a solution). Prove or disprove: there are integers m and n with $6m + 4n = 2$.

Solution

Solution 4. The proposition is true. Take $m = 1$ and $n = -1$. Then $6m + 4n = 6 - 4 = 2$.

Exercise 5 (An equation without a solution). Prove or disprove: there are integers m and n with $6m + 4n = 3$.

Solution

Solution 5. The proposition is false. Applying [Theorem: Negating a quantifier](#) to each of its two quantifiers in turn, its negation is “for all integers m and n , $6m + 4n \neq 3$ ”, and we prove that. Let m and n be integers. Then $6m + 4n = 2(3m + 2n)$, and $3m + 2n$ is an integer ([Remark: Facts about integers used without proof](#)), so $6m + 4n$ is even ([Definition: Even integer](#)). But $3 = 2 \cdot 1 + 1$ is odd ([Definition: Odd integer](#)), and no integer is both even and odd ([Remark: Facts about integers used without proof](#)), so $6m + 4n \neq 3$.

References

Wikipedia contributors. 2026. *Constructive Proof* — *Wikipedia, the Free Encyclopedia*. https://en.wikipedia.org/wiki/Constructive_proof.