Midterm 2 note sheet

Induction, loop invariants, recursion, recurrences and growth

Table 1: **Integers.**

Integers, rationals, reals	$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
Even	$(n \text{ even}) \Leftrightarrow (\exists k \in \mathbb{Z}, n = 2k)$
Odd	$(n \text{ odd}) \Leftrightarrow (\exists k \in \mathbb{Z}, n = 2k + 1)$
Parity	$\forall n \in \mathbb{Z}, (((n \text{ even}) \vee (n \text{ odd})) \wedge (\neg((n \text{ even}) \wedge (n \text{ odd}))))$
Closure	$(a, b \in \mathbb{Z}) \Rightarrow ((a + b \in \mathbb{Z}) \wedge (a - b \in \mathbb{Z}) \wedge (ab \in \mathbb{Z}))$
Divides $(a, b \in \mathbb{Z})$	$(a \mid b) \Leftrightarrow (\exists k \in \mathbb{Z}, b = ak)$
Congruence $(a, b, n \in \mathbb{Z},\ n > 0)$	$(a \equiv b \pmod{n}) \Leftrightarrow (n \mid (a - b))$
Remainder r of a by n $(a, n \in \mathbb{Z},\ n > 0)$	the unique r with $\exists q \in \mathbb{Z}, ((a = qn + r) \wedge (0 \leq r < n))$
Rational	$(x \in \mathbb{Q}) \Leftrightarrow (\exists a, b \in \mathbb{Z}, ((b \neq 0) \wedge (x = a/b)))$

Table 2: **Induction.**

Sum	$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$
Induction $(b, k, n \in \mathbb{Z})$	$(P(b) \wedge (\forall k \geq b, (P(k) \Rightarrow P(k+1)))) \Rightarrow (\forall n \geq b, P(n))$
Base case	$P(b)$
Inductive step	assume $P(k)$ (the induction hypothesis), show $P(k+1)$

Table 3: **Strong induction.**

Strong induction $(b, c, k, n \in \mathbb{Z})$	$(P(b) \wedge \cdots \wedge P(c) \wedge (\forall k \geq c, ((P(b) \wedge \cdots \wedge P(k)) \Rightarrow P(k+1)))) \Rightarrow (\forall n \geq b, P(n))$

Table 4: **Loop invariants.**

Loop invariant I, guard G	I is true each time G is tested
Initialization	I holds when G is first tested
Maintenance	$(I \wedge G)$ before a pass through the body $\Rightarrow$ I after it
Termination	the loop stops, and $(I \wedge (\neg G)) \Rightarrow$ the result

Table 5: **Racket.**

Define a function	`(define (f x y) body)`
Conditionals	`(if test then-expr else-expr)`
	`(cond [test1 expr1] [test2 expr2] [else expr3])`
Lists	`'()` empty, `(list 1 2 3)` gives `'(1 2 3)`
	`(cons x lst)` puts `x` in front, `(empty? lst)` is `#t` if `lst` is `'()`
	`(first lst)`, `(second lst)`, `(third lst)`, `(rest lst)` all but the first
Arithmetic	`(+ a b)` `(- a b)` `(* a b)` `(quotient n d)` `(remainder n d)`
Comparison	`(= a b)` `(< a b)` `(<= a b)` `(> a b)` `(>= a b)`
	`(equal? a b)` is `#t` if `a` and `b` are the same value, lists included
Logic	`(and a b)` `(or a b)` `(not a)`

Table 6: **Recurrences.**

Recurrence relation	each term from earlier terms, starting from initial terms
Iteration	substitute repeatedly, write the result after j steps, choose j to reach an initial term
Geometric sum $(r \neq 1)$	$1 + r + \cdots + r^k = \dfrac{r^{k+1} - 1}{r - 1}$
Arithmetic sum	$1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$
Logarithms	$2^{\log_2 n} = n$
	$\log_2(xy) = \log_2 x + \log_2 y$
	$\log_2(x^k) = k \log_2 x$

Table 7: **Growth of functions.**

Big-O $(c \in \mathbb{R};\ N, n \in \mathbb{Z})$	$(f(n) = O(g(n))) \Leftrightarrow (\exists c > 0, \exists N, \forall n \geq N, \lvert f(n)\rvert \leq c\lvert g(n)\rvert)$
Big-Omega	$(f(n) = \Omega(g(n))) \Leftrightarrow (\exists c > 0, \exists N, \forall n \geq N, \lvert f(n)\rvert \geq c\lvert g(n)\rvert)$
Big-Theta	$(f(n) = \Theta(g(n))) \Leftrightarrow ((f(n) = O(g(n))) \wedge (f(n) = \Omega(g(n))))$
Witnesses	the numbers c and N