Midterm 1 note sheet

Logic, integers, sets and induction

Table 1: **Connectives.**

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Table 2: **Laws.**

Logically equivalent	$A \equiv B$: A and B have the same truth value in every row
Double negation	$(\neg\neg P) \equiv P$
De Morgan	$(\neg(P \wedge Q)) \equiv ((\neg P) \vee (\neg Q))$
	$(\neg(P \vee Q)) \equiv ((\neg P) \wedge (\neg Q))$
Conditional	$(P \Rightarrow Q) \equiv ((\neg P) \vee Q)$
Biconditional	$(P \Leftrightarrow Q) \equiv ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$
Distributive	$(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$
	$(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$
Converse of $P \Rightarrow Q$	$Q \Rightarrow P$
Contrapositive of $P \Rightarrow Q$	$(\neg Q) \Rightarrow (\neg P)$
	$((\neg Q) \Rightarrow (\neg P)) \equiv (P \Rightarrow Q)$

Table 3: **Quantifiers.**

For all	$\forall x \in D, P(x)$
There exists	$\exists x \in D, P(x)$
Negating a quantifier	$(\neg(\forall x \in D, P(x))) \equiv (\exists x \in D, (\neg P(x)))$
	$(\neg(\exists x \in D, P(x))) \equiv (\forall x \in D, (\neg P(x)))$
Counterexample to $\forall x \in D, P(x)$	an $x \in D$ with $\neg P(x)$

Table 4: **Integers.**

Integers, rationals, reals	$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
Even	(n even) $\Leftrightarrow (\exists k \in \mathbb{Z}, n = 2k)$
Odd	(n odd) $\Leftrightarrow (\exists k \in \mathbb{Z}, n = 2k + 1)$
Parity	$\forall n \in \mathbb{Z}, (((n \text{ even}) \vee (n \text{ odd})) \wedge (\neg((n \text{ even}) \wedge (n \text{ odd}))))$
Closure	$(a, b \in \mathbb{Z}) \Rightarrow ((a + b \in \mathbb{Z}) \wedge (a - b \in \mathbb{Z}) \wedge (ab \in \mathbb{Z}))$
Divides $(a, b \in \mathbb{Z})$	$(a \mid b) \Leftrightarrow (\exists k \in \mathbb{Z}, b = ak)$
Congruence $(a, b, n \in \mathbb{Z},\ n > 0)$	$(a \equiv b \pmod n) \Leftrightarrow (n \mid (a - b))$
Remainder r of a by n $(a, n \in \mathbb{Z},\ n > 0)$	the unique r with $\exists q \in \mathbb{Z}, ((a = qn + r) \wedge (0 \leq r < n))$
Rational	$(x \in \mathbb{Q}) \Leftrightarrow (\exists a, b \in \mathbb{Z}, ((b \neq 0) \wedge (x = a/b)))$

Table 5: **Sets.**

Element	$x \in A$
Empty set, cardinality	$\emptyset$, $\lvert A \rvert$
Subset	$(A \subseteq B) \Leftrightarrow (\forall x, ((x \in A) \Rightarrow (x \in B)))$
Set equality	$(A = B) \Leftrightarrow ((A \subseteq B) \wedge (B \subseteq A))$
Union	$A \cup B = \{x : (x \in A) \vee (x \in B)\}$
Intersection	$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$
Difference	$A - B = \{x : (x \in A) \wedge (x \notin B)\}$
Power set	$\mathcal{P}(A) = \{S : S \subseteq A\}$
Cartesian product	$A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}$

Table 6: **Induction.**

Sum	$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$
Induction $(b, k, n \in \mathbb{Z})$	$(P(b) \wedge (\forall k \geq b, (P(k) \Rightarrow P(k+1)))) \Rightarrow (\forall n \geq b, P(n))$
Base case	$P(b)$
Inductive step	assume $P(k)$ (the induction hypothesis), show $P(k+1)$