# Final exam note sheet

The whole course: the tables of the three midterm sheets

Table 1: **Connectives.**

| $P$ | $Q$ | $\neg P$ | $P \wedge Q$ | $P \vee Q$ | $P \Rightarrow Q$ | $P \Leftrightarrow Q$ |
|---|---|---|---|---|---|---|
| T | T | F | T | T | T | T |
| T | F | F | F | T | F | F |
| F | T | T | F | T | T | F |
| F | F | T | F | F | T | T |

Table 2: **Laws.**

| | |
|---|---|
| Logically equivalent | $A \equiv B$: $A$ and $B$ have the same truth value in every row |
| Double negation | $(\neg\neg P) \equiv P$ |
| De Morgan | $(\neg(P \wedge Q)) \equiv ((\neg P) \vee (\neg Q))$ |
| | $(\neg(P \vee Q)) \equiv ((\neg P) \wedge (\neg Q))$ |
| Conditional | $(P \Rightarrow Q) \equiv ((\neg P) \vee Q)$ |
| Biconditional | $(P \Leftrightarrow Q) \equiv ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$ |
| Distributive | $(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$ |
| | $(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$ |
| Converse of $P \Rightarrow Q$ | $Q \Rightarrow P$ |
| Contrapositive of $P \Rightarrow Q$ | $(\neg Q) \Rightarrow (\neg P)$ |
| | $((\neg Q) \Rightarrow (\neg P)) \equiv (P \Rightarrow Q)$ |

Table 3: **Quantifiers.**

| | |
|---|---|
| For all | $\forall x \in D, P(x)$ |
| There exists | $\exists x \in D, P(x)$ |
| Negating a quantifier | $(\neg(\forall x \in D, P(x))) \equiv (\exists x \in D, (\neg P(x)))$ |
| | $(\neg(\exists x \in D, P(x))) \equiv (\forall x \in D, (\neg P(x)))$ |
| Counterexample to $\forall x \in D, P(x)$ | an $x \in D$ with $\neg P(x)$ |

Table 4: **Integers.**

| | |
|---|---|
| Integers, rationals, reals | $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ |
| Even | ($n$ even) $\Leftrightarrow (\exists k \in \mathbb{Z}, n = 2k)$ |
| Odd | ($n$ odd) $\Leftrightarrow (\exists k \in \mathbb{Z}, n = 2k+1)$ |
| Parity | $\forall n \in \mathbb{Z}, ((n \text{ even}) \vee (n \text{ odd})) \wedge (\neg((n \text{ even}) \wedge (n \text{ odd}))))$ |
| Closure | $(a, b \in \mathbb{Z}) \Rightarrow ((a+b \in \mathbb{Z}) \wedge (a-b \in \mathbb{Z}) \wedge (ab \in \mathbb{Z}))$ |
| Divides $(a, b \in \mathbb{Z})$ | $(a \mid b) \Leftrightarrow (\exists k \in \mathbb{Z}, b = ak)$ |
| Congruence $(a, b, n \in \mathbb{Z},\ n > 0)$ | $(a \equiv b \pmod n) \Leftrightarrow (n \mid (a-b))$ |
| Remainder $r$ of $a$ by $n$ $(a, n \in \mathbb{Z},\ n > 0)$ | the unique $r$ with $\exists q \in \mathbb{Z}, ((a = qn + r) \wedge (0 \le r < n))$ |
| Rational | $(x \in \mathbb{Q}) \Leftrightarrow (\exists a, b \in \mathbb{Z}, ((b \neq 0) \wedge (x = a/b)))$ |

Table 5: **Sets.**

| | |
|---|---|
| Element | $x \in A$ |
| Empty set, cardinality | $\emptyset$, $\lvert A \rvert$ |
| Subset | $(A \subseteq B) \Leftrightarrow (\forall x, ((x \in A) \Rightarrow (x \in B)))$ |
| Set equality | $(A = B) \Leftrightarrow ((A \subseteq B) \wedge (B \subseteq A))$ |
| Union | $A \cup B = \{x : (x \in A) \vee (x \in B)\}$ |
| Intersection | $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$ |
| Difference | $A - B = \{x : (x \in A) \wedge (x \notin B)\}$ |
| Power set | $\mathcal{P}(A) = \{S : S \subseteq A\}$ |
| Cartesian product | $A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}$ |

Table 6: **Induction.**

| | |
|---|---|
| Sum | $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ |
| Induction $(b, k, n \in \mathbb{Z})$ | $(P(b) \wedge (\forall k \ge b, (P(k) \Rightarrow P(k+1)))) \Rightarrow (\forall n \ge b, P(n))$ |
| Base case | $P(b)$ |
| Inductive step | assume $P(k)$ (the induction hypothesis), show $P(k+1)$ |

Table 7: **Strong induction.**

| | |
|---|---|
| Strong induction $(b, c, k, n \in \mathbb{Z})$ | $(P(b) \wedge \cdots \wedge P(c) \wedge (\forall k \ge c, ((P(b) \wedge \cdots \wedge P(k)) \Rightarrow P(k+1)))) \Rightarrow (\forall n \ge b, P(n))$ |

Table 8: **Loop invariants.**

| | |
|---|---|
| Loop invariant $I$, guard $G$ | $I$ is true each time $G$ is tested |
| Initialization | $I$ holds when $G$ is first tested |
| Maintenance | $(I \wedge G)$ before a pass through the body $\Rightarrow I$ after it |
| Termination | the loop stops, and $(I \wedge (\neg G)) \Rightarrow$ the result |

Table 9: **Racket.**

| | |
|---|---|
| Define a function | `(define (f x y) body)` |
| Conditionals | `(if test then-expr else-expr)` |
| | `(cond [test1 expr1] [test2 expr2] [else expr3])` |
| Lists | `'()` empty, `(list 1 2 3)` gives `'(1 2 3)` |
| | `(cons x lst)` puts `x` in front, `(empty? lst)` is `#t` if `lst` is `'()` |
| | `(first lst)`, `(second lst)`, `(third lst)`, `(rest lst)` all but the first |
| Arithmetic | `(+ a b)` `(- a b)` `(* a b)` `(quotient n d)` `(remainder n d)` |
| Comparison | `(= a b)` `(< a b)` `(<= a b)` `(> a b)` `(>= a b)` |
| | `(equal? a b)` is `#t` if `a` and `b` are the same value, lists included |
| Logic | `(and a b)` `(or a b)` `(not a)` |

Table 10: **Recurrences.**

| | |
|---|---|
| Recurrence relation | each term from earlier terms, starting from initial terms |
| Iteration | substitute repeatedly, write the result after $j$ steps, choose $j$ to reach an initial term |
| Geometric sum $(r \neq 1)$ | $1 + r + \cdots + r^k = \dfrac{r^{k+1} - 1}{r - 1}$ |
| Arithmetic sum | $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ |
| Logarithms | $2^{\log_2 n} = n$ |
| | $\log_2(xy) = \log_2 x + \log_2 y$ |
| | $\log_2(x^k) = k \log_2 x$ |

Table 11: **Growth of functions.**

| | |
|---|---|
| Big-O $(c \in \mathbb{R};\ N, n \in \mathbb{Z})$ | $(f(n) = O(g(n))) \Leftrightarrow (\exists c > 0, \exists N, \forall n \ge N, \lvert f(n) \rvert \le c \lvert g(n) \rvert)$ |
| Big-Omega | $(f(n) = \Omega(g(n))) \Leftrightarrow (\exists c > 0, \exists N, \forall n \ge N, \lvert f(n) \rvert \ge c \lvert g(n) \rvert)$ |
| Big-Theta | $(f(n) = \Theta(g(n))) \Leftrightarrow ((f(n) = O(g(n))) \wedge (f(n) = \Omega(g(n))))$ |
| Witnesses | the numbers $c$ and $N$ |

Table 12: **Relations and functions.**

| | |
|---|---|
| Relation on $A$ | $R \subseteq A \times A$, and $(a\ R\ b) \Leftrightarrow ((a,b) \in R)$ |
| Reflexive | $\forall a \in A, a\ R\ a$ |
| Symmetric | $\forall a, b \in A, ((a\ R\ b) \Rightarrow (b\ R\ a))$ |
| Transitive | $\forall a, b, c \in A, (((a\ R\ b) \wedge (b\ R\ c)) \Rightarrow (a\ R\ c))$ |
| Antisymmetric | $\forall a, b \in A, (((a\ R\ b) \wedge (b\ R\ a)) \Rightarrow (a = b))$ |
| Equivalence relation | reflexive, symmetric and transitive |
| Equivalence class | $[a] = \{b \in A : b\ R\ a\}$ |
| Injective $f : A \to B$ | $\forall a_1, a_2 \in A, ((f(a_1) = f(a_2)) \Rightarrow (a_1 = a_2))$ |
| Surjective | $\forall b \in B, \exists a \in A, f(a) = b$ |
| Bijective | injective and surjective |
| Inverse of a bijection | $f^{-1} : B \to A$ with $(f^{-1}(b) = a) \Leftrightarrow (f(a) = b)$ |
| Floor | $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ |

Table 13: **Graphs.**

| | |
|---|---|
| Degree | $\deg(v)$: edges with endpoint $v$, a loop counting twice |
| Walk from $u$ to $v$ | $v_0, e_1, v_1, \ldots, e_n, v_n$, $v_0 = u$, $v_n = v$, $e_i$ joins $v_{i-1}$ and $v_i$; length $n$ |
| Trail, path | a walk with no repeated edge; with no repeated vertex |
| Circuit | a trail of length $\geq 1$ from a vertex to itself |
| Connected | a walk between every two vertices |
| Euler circuit | a circuit that contains every edge |
| Euler trail | a trail from $u$ to $v \neq u$ that contains every edge |
| Hamiltonian cycle | a circuit that visits every vertex once, then returns to its start |
| Bipartite | $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, and every edge has one endpoint in $V_1$ and one in $V_2$ |

```
procedure bfs(G, s)                      procedure dfs(G, v)   // start: dfs(G, s)
    mark s, and set dist[s] := 0             mark v
    Q := a queue containing only s           for each vertex u in the list of v, in order
    while Q is not empty                         if u is not marked
        v := remove the front of Q                   dfs(G, u)
        for each vertex u in the list of v, in order
            if u is not marked
                mark u, and set dist[u] := dist[v] + 1
                add u to the back of Q
```

Table 14: **Finite automata.**

| | |
|---|---|
| DFA | $M = (S, \Sigma, \delta, s_0, F)$: states, alphabet, $\delta : S \times \Sigma \to S$, initial state, accepting states |
| Transition table | $\to$ marks the initial state, $*$ each accepting state |
| Empty string | $\lambda$ |
| Run on $w = x_1 \cdots x_n$ | $r_0, \ldots, r_n$ with $r_0 = s_0$ and $r_i = \delta(r_{i-1}, x_i)$ |
| Accepts $w$ | $r_n \in F$ |
| Language | $L(M) = \{w : M \text{ accepts } w\}$ |

Table 15: **Trees.**

| | |
|---|---|
| Tree | a connected graph with no cycles |
| Edges of a tree | a tree with $n$ vertices has $n - 1$ edges |
| Binary tree | `nil`, or node$(x, L, R)$: root label $x$, subtrees $L$ and $R$ |
| In Racket | `nil` is `'()`, node$(x, L, R)$ is `(list x L R)` |
| Preorder | root, preorder of $L$, preorder of $R$ |
| Inorder | inorder of $L$, root, inorder of $R$ |
| Postorder | postorder of $L$, postorder of $R$, root |
| Level, height | the root is at level 0, a child one below its parent; height is the largest level |
| Leaf | a vertex with no children |
| Leaf of a tree with no root | a vertex of degree 1 |